WHAT IS A PHOTON?

Spontaneous emission: The need for quantum field theory

In these notes I would like to try and give an introduction to the quantum mechanical theory of the photon. The treatment I give is in the spirit of a treatment you can find in Dirac's quantum mechanics monograph, *The Principles of Quantum Mechanics*. I believe that Dirac was one of the first (if not *the* first) person to work out these ideas. Along the way, we will be generalizing the way we use quantum mechanics in a non-trivial way. More precisely, the way we model nature using the rules of quantum mechanics will change significantly, although the rules themselves will not actually change. Let us begin by setting the stage for this generalization.

Consider the well-known processes of emission and absorption of photons by atoms. The processes of emission and absorption of photons by atoms cannot, ultimately, be accommodated in the usual quantum mechanical models based on particle mechanics. Instead, one must use a new class of models that go under the heading of *quantum field theory*. The reasons for this necessity are relatively simple if one focuses on *spontaneous emission* in atoms. This is where an atom in an excited state will spontaneously decay to a lower energy state (and emit one or more photons). First, we all know that the electron states we use to characterize atoms are the *stationary states*, which are energy eigenstates. But stationary states have the property that all their observable features are time independent. If an atomic electron occupying an atomic energy level were truly in a stationary state there could never be any spontaneous emission since a stationary state has no time dependent behavior. The only way out of this conundrum is to suppose that atomic energy levels are not really stationary states once you take into account the interaction of photons and electrons. But now consider emission of a photon by an atomic electron. The initial state of the system has an electron. The final state of the system has an electron and a photon. Now, in the usual quantum mechanical formalism for particles the number of particles is always fixed. Indeed, the normalization of the wave function for a particle (or for particles) can be viewed as saying that the particle (or particles) is (or are) always somewhere. Evidently, such a state of affairs will not allow us to treat a particle such as a photon that can appear and disappear. Moreover, it is possible to have atomic transitions in which more than one photon appears/disappears. Clearly we will not be able to describe such processes using the quantum mechanical models developed thus far. If this isn't surprising enough, I remind you that there exist situations in which a photon may "transform" into an electron-positron pair and, conversely, in which an electron-positron can turn into a photon. So even electrons are not immune from the appearance/disappearance phenomenon.

Remarkably, it is possible to describe these multi-particle processes using the axioms of quantum theory, provided these axioms are used in a clever enough way. This new and improved use of quantum mechanics is usually called *quantum field theory* since it can be viewed as an application of the basic axioms of quantum mechanics to continuous systems ("field theories") rather than mechanical systems. The picture that emerges is that the building blocks of matter and its interactions are neither particles nor waves, but a new kind of entity: a quantum field. Every type of elementary particle is described by a quantum field (although the groupings by type depend upon the sophistication of the model). There is an electron-positron field, a photon field, a neutrino field, and so forth. In this way of doing things, particles are elementary excitations of the quantum field.

Quantum field theory (QFT) has lead to spectacular successes in describing the behavior of a wide variety of atomic and subatomic phenomena. The success is not just qualitative; some of the most precise measurements known involve minute properties of the spectra of atoms. These properties are predicted via quantum field theory and, so far, the agreement with experiment is perfect.

We will only be able to give a very brief, very superficial, very crude introduction to some of the basic ideas. Our goal will be to show how QFT is used to describe photons. In a future discussion this could be the basis of an explanation of phenomena such as spontaneous emission.

Harmonic Oscillators again

Our first goal will be to describe the photon without considering its interaction with other (charged) particles. This is a quantum version of considering the EM field in the absence of sources. Indeed, our strategy for describing photons will be to extract them from a "quantization" of the source-free electromagnetic field. The key idea that allows this point of view is that the EM field can, in the absence of sources, be viewed as an infinite collection of coupled harmonic oscillators. We know how to describe harmonic oscillators quantum mechanically, and we can try to carry this information over to the EM field. First, let us quickly review the key properties of the harmonic oscillator.

Recall that the energy of a classical oscillator with displacement $x(t)$ is given by

$$E = \frac{m}{2}\dot{x}^2 + \frac{1}{2}m\omega^2 x^2.$$

In the quantum description, we view the energy in terms of coordinate and momentum operators, in which case the energy is known as the Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Here we view x and p as operators on a Hilbert space obeying the commutation relation

$$[x, p] = i\hbar\mathbf{1},$$

with "$\mathbf{1}$" being the identity operator. Let us recall the definition of the "ladder operators":

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + i\frac{p}{m\omega}), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - i\frac{p}{m\omega}),$$

satisfying the commutation relations (exercise)

$$[a, a^\dagger] = \mathbf{1}.$$

The Hamiltonian takes the form (exercise)

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}\mathbf{1}).$$

We can drop the the second term (with the "$\frac{1}{2}$") if we want; it just defines the zero point of energy. The stationary states are labeled by a non-negative integer n,

$$H|n\rangle = E_n|n\rangle, \quad E_n = (n + \frac{1}{2})\hbar\omega.$$

The ground state $|0\rangle$ satisfies

$$a|0\rangle = 0.$$

Excited states are obtained via the identity (exercise)

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

We also have

$$a|n\rangle = \sqrt{n}|n-1\rangle.$$

It is easy to generalize this treatment to a system consisting of a number of uncoupled harmonic oscillators with displacements x_i, $i = 1, 2, \ldots$, momenta p_i, masses m_i and frequencies ω_i. In particular, the Hamiltonian is (exercise)

$$H = \sum_i \frac{1}{2}\left(\frac{p_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2 x_i^2\right) = \sum_i \left(a_i^\dagger a_i + \frac{1}{2}\mathbf{1}\right).$$

Even with a set of coupled oscillators, if the couplings are themselves harmonic, we can pass to the normal mode coordinates and momenta. In this case the Hamiltonian again takes the form given above. So, this description is quite general.

Fourier components of an EM field

To find a quantum description of the EM field we need some elementary results from classical electromagnetic theory. To begin with, we introduce the EM potentials. Recall that the homogeneous subset of the Maxwell equations

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

and

$$\nabla \cdot \mathbf{B} = 0$$

are equivalent to the existence of a vector field, the *vector potential* $\mathbf{A}$, and a scalar field, the *scalar potential* ϕ, such that

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

This is the general solution to the homogeneous subset of the Maxwell equations, so if we choose to work with the electromagnetic potentials we have eliminated half of the Maxwell equations.

The potentials are far from uniquely defined. If $(\phi, \mathbf{A})$ are a set of potentials for a given EM field $(\mathbf{E}, \mathbf{B})$, then so are (exercise)

$$\phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t},$$

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda,$$

where Λ is *any* (well-behaved) function of space and time. This transformation between two sets of potentials for the same EM field is called a *gauge transformation*, for historical reasons that we shall not go into. Physical quantities will be unchanged by gauge transformations. The notion of gauge invariance, which just seems like a technical detail in Maxwell theory, is actually pretty profound in physics. However, for our purposes, we just note that the freedom to redefine potentials via a gauge transformation means that we can try to make a convenient choice of potentials. Our choice will be always to put the potentials in the *radiation gauge*. What this means is as follows. Any electromagnetic field can be described by a set of potentials such that

$$\phi = 0, \quad \nabla \cdot \mathbf{A} = 0.$$

This should amuse you (at least a little). In electrostatics it is conventional to work in a gauge in which $\mathbf{A} = 0$ and then the static electric field is (minus) the gradient of the scalar potential. This is certainly the most convenient way to analyze electrostatics, but one *could* opt to use a time-dependent (and curl-free) vector potential if so-desired (exercise).

The Hamiltonian of the electromagnetic field

To use the harmonic oscillator paradigm to "quantize" the EM field, we first express the total energy of the field in terms of the potentials:

$$H = \frac{1}{8\pi}\int_{\text{all space}} d^3x\,(E^2+B^2) = \frac{1}{8\pi}\int_{\text{all space}} d^3x\,\left[\frac{1}{c^2}\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 + (\nabla\times\mathbf{A})^2\right].$$

You can think of the first integral in the sum as the kinetic energy of the field and the second integral as the potential energy. This is something more than an analogy. It is possible to think of the electromagnetic field as a Hamiltonian dynamical system with the vector potential playing the role of (an infinite number of) generalized coordinate(s) and the electric field as its "canonically conjugate momentum". In this interpretation the function H above is the Hamiltonian (expressed in terms of position and velocity). The behavior of any such dynamical system is determined by the Hamiltonian expressed as a function of canonical coordinates and momenta, therefore we focus all our attention on H.

Next we make a Fourier decomposition of $\mathbf{A}$:

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,\mathbf{A}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Note that (exercises)

$$(A_{\mathbf{k}})^* = A_{-\mathbf{k}},$$

$$\nabla\cdot\mathbf{A} = 0 \Longleftrightarrow \mathbf{k}\cdot\mathbf{A}_{\mathbf{k}} = 0,$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{(2\pi)^{3/2}}\int d^3k\,\dot{\mathbf{A}}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\nabla\times\mathbf{A} = \frac{i}{(2\pi)^{3/2}}\int d^3k\,[\mathbf{k}\times\mathbf{A}_{\mathbf{k}}(t)]e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Insert the Fourier expansion of the vector potential into the Hamiltonian. We get (exercise):

$$H = \int d^3k\left\{\frac{1}{c^2}|\dot{\mathbf{A}}_{\mathbf{k}}|^2 + |\mathbf{k}\times\mathbf{A}_{\mathbf{k}}|^2\right\}$$

Using (1) a vector identity for the dot product of a pair of cross products, and (2) the radiation gauge condition, we get:

$$H = \int d^3k\left\{\frac{1}{c^2}|\dot{\mathbf{A}}_{\mathbf{k}}|^2 + k^2|A_{\mathbf{k}}|^2\right\}.$$

Let's compare this with the harmonic oscillator Hamiltonian. Think of the integrand as the energy of a single oscillator and the integral as a sum. Then we have (exercise)

$$\omega \longrightarrow \omega(k) = kc,$$

and

$$m \longrightarrow \frac{1}{c^2}.$$

The frequency correspondence is perfectly reasonable; it describes the frequency-wave number dispersion relation of an EM wave. The mass analogy is okay mathematically, but shouldn't be taken too literally in a physical sense; there is no particularly meaningful way to ascribe a rest mass to the electromagnetic field.

Further confirmation of our interpretation of things comes by considering the remaining Maxwell equations,*

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0$$

and

$$\nabla \cdot \mathbf{E} = 0,$$

which imply that each component of $\mathbf{A}$ satisfies the wave equation with propagation velocity c (and we still have the side condition $\nabla \cdot \mathbf{A} = 0$). This means that (exercise)

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{c}_{\mathbf{k}} e^{i\omega(k)t} + \mathbf{c}^*_{-\mathbf{k}} e^{-i\omega(k)t},$$

for some constants $\mathbf{c}_{\mathbf{k}}$. Thus for each value of $\mathbf{k}$ the vector potential behaves like 3 harmonic oscillators with frequency kc.

Thus you can think of the vector potential as a sort of generalized coordinate, and the electric field as the canonically conjugate momentum. The Hamiltonian is then that of a (continuous) collection of harmonic oscillators labeled by the wave vector of a Fourier decomposition. The fact that there is a continuous family of coordinates and momenta (one for each spatial point, or one for each wave vector) leads to the statement that the EM field has "an infinite number of degrees of freedom". This is the principal feature distinguishing quantum field theory from quantum mechanics, and it is what allows one to describe processes in which particles are created and destroyed.

Now let us massage our formula for the Hamiltonian into an even nicer form. We do this by taking account of the properties of the Fourier components of the vector potential. We have seen that the Fourier component of the vector potential with wave vector $\mathbf{k}$ is orthogonal to $\mathbf{k}$, and that it satisfies a complex conjugation relation. We take both of these conditions into account (and introduce a convenient normalization) via the definition

$$\mathbf{A}_{\mathbf{k}} = \sum_{\sigma=1}^{2} \sqrt{\frac{\hbar c}{k}} (a_{\mathbf{k},\sigma} \epsilon_{\mathbf{k},\sigma} + a^*_{-\mathbf{k},\sigma} \epsilon_{-\mathbf{k},\sigma}),$$

* Recall that we consider the electromagnetic field in regions of space in which there are no charges or currents.

where σ labels the polarization and $\epsilon_{\mathbf{k},\sigma}$ are two orthonormal vectors orthogonal to $\mathbf{k}$. The variables $a_{\mathbf{k},\sigma}$ carry the amplitude information about each polarization, and the polarization direction is determined by the choice of $\epsilon_{\mathbf{k},\sigma}$. Note that we have used $\hbar$ to define the new variables. From a purely classical EM point of view this is a bit strange, but it is convenient for the quantum treatment we will give. For now, just think of the use of $\hbar$ as a convenient way of forming the amplitudes $a_{\mathbf{k},\sigma}$, which are dimensionless (exercise). Using this form of $\mathbf{A}_{\mathbf{k}}(t)$ it follows that, for solutions of the Maxwell equations,

$$\frac{d}{dt}\mathbf{A}_{\mathbf{k}} = i\sqrt{\hbar k}c\sum_{\sigma=1}^{2}(a_{\mathbf{k},\sigma}\epsilon_{\mathbf{k},\sigma} - a^*_{-\mathbf{k},\sigma}\epsilon_{\mathbf{k},\sigma}).$$

Using this result we have (exercise)

$$H = \int d^3k \sum_\sigma \hbar\omega(k)\left(a^*_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right).$$

Hopefully, this very simple form for the energy justifies to you all the effort that went into deriving it. Up to a choice of zero point of energy, this is clearly a classical version of the Hamiltonian for a collection of oscillators labeled by $\mathbf{k}$ and σ. A similar computation with the Lagrangian for the Maxwell field,

$$L = \frac{1}{8\pi}\int_{\text{all space}} d^3x(E^2 - B^2),$$

leads to a sum (really, integral) of harmonic oscillator Lagrangians. Thus, mathematically at least, the electromagnetic field (in the radiation gauge) *is* a continuous collection of harmonic oscillators.

The quantization of the EM field

The classical Hamiltonian for the electromagnetic field can be expressed as a continuous superposition over harmonic oscillator Hamiltonians:

$$H = \int d^3k \sum_\sigma \hbar\omega(k)\left(a^*_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right).$$

We thus view the quantum EM field as an infinite set of quantum oscillators. The oscillators' degrees of freedom are labeled by the wave vector $\mathbf{k}$ and the polarization σ. We view the ladder operators for each degree of freedom as $a_{\mathbf{k},\sigma}$ and $a^\dagger_{\mathbf{k},\sigma}$. In the context of quantum field theory, we call these operators *annihilation* and *creation* operators, respectively. We will see why in a moment. The quantum Hamiltonian is built from the creation and annihilation operators via

$$H = \int d^3k \sum_\sigma \hbar\omega(k)\left(a^\dagger_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right).$$

This is clearly just the sum of energies for each individual oscillator (with the "zero point energy" dropped).

Incidentally, it is not too hard to compute the total momentum $\mathbf{P}$ of the electromagnetic field. It is obtained from the integral of the Poynting vector. (This means that the Cartesian components of the total momentum are integrals of the corresponding components of the Poynting vector field. In terms of the creation and annihilation operators we get (exercise)

$$\mathbf{P} = \frac{c}{4\pi}\int d^3x\, \mathbf{E}\times\mathbf{B} = \int d^3k \sum_\sigma \hbar\mathbf{k}\left(a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma}\right).$$

This is the same as the total energy except that the energy of each mode, $\hbar\omega(k)$ has been replaced by the momentum of each mode $\hbar\mathbf{k}$.

Do you recall the usual lore of photons? You know, the lore that says a photon with a definite energy and momentum will have $E = \hbar\omega$, and $\mathbf{P} = \hbar\mathbf{k}$, where $\omega = kc$. We see that we are in a position to model a photon with definite energy and momentum as a quantum normal mode of the EM field!

To make detailed sense of all this we should spell out the meaning of all these ladder operators. The idea is to first use the harmonic oscillator point of view to understand the definition of the operators. Then we can reinterpret the mathematical set-up in terms of photon states.

The vacuum

To begin with, let us suppose that all the oscillators are in their ground state. This state of the EM field, denoted $|0\rangle$, is called the *vacuum state*. You can easily see why. This state is an eigenstate of H and $\mathbf{P}$ with eigenvalue zero:

$$H|0\rangle = 0 = \mathbf{P}|0\rangle.$$

This you can verify from the fact that all the annihilation (*i.e.,* lowering) operators map the ground state to the zero vector:

$$a_{\mathbf{k},\sigma}|0\rangle = 0.$$

Thus the vacuum of the EM field (in the absence of any other interactions, which we are not modeling here) is a state in which the energy and momentum are known with probability one. It can be shown that this state is the state of lowest energy and momentum.

By the way, have you ever encountered claims that, thanks to the uncertainty principle, there is an "infinite reservoir of zero point energy in the vacuum"? Perhaps you have even seen seemingly learned schemes to extract this energy for practical use. Now you are in a position to be a little skeptical: the total energy-momentum of the vacuum is perfectly well-defined – it vanishes! Where do these wacky claims come from? As with most popular distortions of scientific results, there is a kernel of truth here. To uncover it, return to the case of a single harmonic oscillator. The ground state energy is perfectly well defined, but the energy and position and/or momentum operators are not compatible. This means that if you know the energy with probability one, you cannot know the position or momentum of the oscillator with probability one. Indeed, you may recall that in the ground state of the oscillator the probability distributions for position and momentum are Gaussians with zero average. The EM field has a similar behavior. Recall that the "position" is, essentially, the vector potential (hence the magnetic field is a "function of position") and the "momentum" is, essentially, the electric field. In the EM vacuum state, the energy is minimized and sharply defined, but the EM fields themselves "fluctuate". More precisely, the EM fields have a probability distribution with non-zero standard deviation; basically, each mode (and polarization) is a described by a Gaussian probability distribution in the vacuum state. (This fact is responsible for the "Casimir effect".) Very roughly speaking, the uncertainty principle means that, while the total energy is known with certainty, the energy density is "uncertain". I say "very roughly" since the notion of quantum energy density of an EM field is rather touchy; there is no well-defined quantum version that behaves much like the classical analog. Indeed, much of the zero point energy nonsense that appears in print is based upon trying to use classical ideas to describe a feature of the theory that is, well, not at all classical!

Photon states

With the vacuum state under control, we can now consider *excited stationary states of the quantum EM field*, which we will interpret as states with one or more photons. Think again about the EM field as a large collection of harmonic oscillators, labeled by wave number and polarization. Suppose we put one of these oscillators in its first excited state, say, by applying $a^\dagger_{\mathbf{k},\sigma}$ to the vacuum for some fixed choice of $\mathbf{k}$ and σ. We denote this state as

$$|1_{\mathbf{k},\sigma}\rangle = a^\dagger_{\mathbf{k},\sigma}|0\rangle.$$

It can now be shown that the resulting state is an eigenvector of H and $\mathbf{P}$:

$$H|1_{\mathbf{k},\sigma}\rangle = \hbar\omega(k)|1_{\mathbf{k},\sigma}\rangle,$$

$$\mathbf{P}|1_{\mathbf{k},\sigma}\rangle = \hbar\mathbf{k}|1_{\mathbf{k},\sigma}\rangle.$$

This state has energy-momentum values defined with probability unity, which take the form appropriate for a single photon. We interpret this state as a *1 photon state* with the indicated wave number, frequency, and polarization. Thus, the first excited stationary state of the quantum electromagnetic field can be viewed as a single photon. In this sense photons are "quanta of the electromagnetic field". Superpositions of photon states over momenta lead to photons that have probability distributions for their energy and momenta. Thus photons need not have a definite momentum or energy any more than, say, an electron must.

More generally, we can build up a Hilbert space of states by repeatedly applying to the vacuum state the creation operators with various wave numbers and polarizations, taking superpositions, *etc.* The result of each application of the creation operator $a^\dagger_{\mathbf{k},\sigma}$ is to define a state with one more photon of the indicated type. Each application of the annihilation operator $a_{\mathbf{k},\sigma}$ results in a state with a photon of the indicated type removed. If the state doesn't have such a photon, (*e.g.,* the vacuum state won't) then the result is the zero vector.

The states we are describing have a particle interpretation, but the theory is richer than just a collection of particles since, *e.g.,* one can superimpose the states described above to get states which encode the field like properties of the EM field. Of course, such states will not be stationary states. Also, recall our discussion of the "fluctuations" of $\mathbf{E}$ and $\mathbf{B}$ in the vacuum. Charges respond to the electromagnetic field strengths (think: Lorentz force law), and so the behavior of charged particles can be affected by EM phenomena, even when no photons are present! This is the idea behind the "Lamb shift" found in the spectra of atoms. And it is the key idea needed to explain spontaneous emission of photons from atoms.

We see, then, how the "normal modes" of a field satisfying wave-type equations can be "quantized". The resulting theory admits particle-like properties in its stationary states. To date, every known elementary particle has been successfully described by a quantum field in much the way we described photons using a quantized electromagnetic field. Interactions between particles (better: between quantum fields) have also been described with considerable success using the quantum field formalism. Indeed, it is reasonable to adopt the point of view that, in our current best understanding of nature, quantum fields are the stuff out of which everything is made!