Physics 3550

Lagrangian Mechanics

Relevant Sections in Text: Chapters 6 and 7

The Lagrangian formulation of Mechanics – motivation

Some 100 years after Newton devised classical mechanics Euler and Lagrange gave a different, considerably more general way to view dynamics. The ideas underlying their approach go back to Leibniz. The key new idea in this approach is the use of *variational principles* to formulate the equations of motion. One of my goals in this part of the course is to get you acquainted with variational principles, which are useful in a variety of applications beyond mechanics. As far as I can tell, the utility of a variational principle in classical mechanics is not at all obvious and somewhat mysterious – until one appeals to quantum mechanics. It is remarkable that people like Lagrange were able to do what they did long before quantum mechanics was discovered. A proper quantum mechanical explanation for the existence of variational principles for classical mechanics is way beyond the scope of this course, of course. But it's so interesting I can't resist telling you just a little bit about it.

First I need to tell you about Feynman's *path integral formalism* for quantum mechanics. This is a very different way to view quantum mechanics compared to the more traditional formalism of wave functions and the Schrödinger equation. For simplicity, let us just think about the dynamics of a single particle in 3-d. Very roughly speaking, in the path integral formalism one shows that the probability amplitude* for a particle to move from one place, $\vec{r}_1$, to another, $\vec{r}_2$, is given by adding up the probability amplitudes for all possible paths connecting these two positions (not just the classically allowed trajectory). The amplitude for a given path $\vec{r}(t)$ is of the form $e^{\frac{i}{\hbar}S[\vec{r}]}$, where $S[\vec{r}]$ is the *action functional* for the trajectory.† The action functional assigns a number to each path $\vec{r}(t)$ connecting $\vec{r}_1$ to $\vec{r}_2$. The specific way in which the action assigns numbers to paths depends upon the physics (degrees of freedom, masses, potentials, *etc.*) of the system being considered. We shall see examples of action functionals soon. In a classical limit (usually when various parameters characterizing the system are in some sense "macroscopic") it can be shown that the dominant paths in the sum over paths come from critical (or "stationary") points of the action functional. These are paths which have the property that "nearby" paths do not change $S[\vec{r}]$ appreciably. Finding such paths is the essence of a variational principle – we vary the paths until a critical point is found, usually a minimum. The critical points

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* A probability amplitude is a complex number whose absolute value-squared gives a probability.
† A function is a rule which assigns numbers to one or more variables. A function*al* assigns numbers to one or more *functions*.

of the action are the classically allowed paths; we see that the derivation of classical equations of motion from variational principles is preordained by quantum mechanics. This is a satisfying state of affairs given the fact that classical mechanics can be viewed as a macroscopic approximation to quantum mechanics.

Of course, the variational principles of mechanics (19th century) came much earlier than quantum mechanics (1920's), let alone Feynman's path integral approach (1940's). This is a testament to the great minds (Euler, Lagrange, Hamilton, Jacobi, . . .) that found these variational principles! These principles came into favor because they provide a very powerful way to organize information about a dynamical system. In particular, using a single quantity (the Lagrangian or the Hamiltonian) one can deduce (in principle) essentially all aspects of a dynamical system, *e.g.,* equations of motion, symmetries, conservation laws, . . . , even the basic strategy for building the associated quantum system. In fact, modern approaches to modeling dynamical systems take the variational principle as fundamental: we begin by building the Lagrangian or Hamiltonian for the system. As mentioned before, one can think of the discovery of the variational principles of mechanics as really a discovery of a footprint left by the quantum world on the macroscopic world.

We will begin by simply positing the Lagrangian formalism, much as Newton posited his laws of motion. Later I will show you how this formalism arises from a variational principle. Aside from the preceding comments, we will have to leave the derivation of the variational principle from quantum mechanics to more advanced class.

The Lagrangian and the Euler-Lagrange Equations

Newtonian mechanics is based upon the notion of force, which is a vector field you have learned to compute/use in various situations. Lagrangian mechanics is based upon a simpler, more curious quantity: the scalar you get by taking the difference between kinetic (T) and potential V energies. This scalar is the *Lagrangian*,

$$L = T - V.$$

We have certainly learned how to compute T and V, so computing L is no problem. What do we do with it?

Let us begin by considering a particle of mass m moving in one dimension with potential energy V. If we denote the possible positions at any given time by x and the possible velocities at that time by $\dot{x}$, we can write the Lagrangian as:

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x).$$

Here we view the Lagrangian as a function of 2 variables as in ordinary calculus. You should not view x or $\dot{x}$ as functions of time at this point. The prescription of Euler and

Lagrange is that the motion of the particle is determined by a differential equation, and that this differential equation can always be found by forming the *Euler-Lagrange equation*:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0.$$

There's a lot going on here which we will have to nail down. But, for now, let's follow our noses and see what this formula can do for us in our example. We have

$$\frac{\partial L}{\partial x} = -\frac{dV(x)}{dx}, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}.$$

Evidently,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x},$$

so that the Euler-Lagrange (EL) equation is

$$-\frac{dV(x)}{dx} + m\ddot{x} = 0,$$

which is just $F = ma$ for this system.

You can think of the Lagrangian as a function which, via the Euler-Lagrange equation, is used to build equations of motion. When the Lagrangian takes the form $T - V$ (in an inertial reference frame) the equations you get correspond to Newton's second law. As we shall see, Lagrangian mechanics generalizes Newtonian mechanics in a number of useful ways.

Example: The roller coaster (again).

We used the roller coaster before as an example of a one-dimensional dynamical system whose kinetic energy depends upon position. As you will recall, the track of the roller coaster had the graph $y = f(x)$, for some function f. I invite you to compute the equations of motion of the roller coaster (for any given function f) using Newton's laws. Our aim is to compute the equation of motion via the EL equation, which is considerably easier.

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(1 + f'^2(x))\dot{x}^2.$$

Here we used the fact that

$$\frac{d}{dt}f(x(t)) = f'(x(t))\frac{dx(t)}{dt}.$$

The potential energy is

$$V = mgy = mgf(x).$$

3

The Lagrangian is

$$L(x, \dot{x}) = T - V = \frac{1}{2}m(1 + f'^2(x))\dot{x}^2 - mgf(x).$$

We next compute

$$\frac{\partial L}{\partial x} = mf'f''\dot{x}^2 - mgf', \quad \frac{\partial L}{\partial \dot{x}} = m(1 + f'^2)\dot{x}.$$

The equation of motion of the roller coaster is then, after a tiny bit of algebra,

$$(1 + f'^2)\ddot{x} + f'f''\dot{x}^2 + gf' = 0.$$

The complexity of this equation reflects the fact that (1) only the component of gravity (instantaneously) tangent to the roller coaster track can cause an acceleration, and (2) we are using the horizontal displacement x to characterize the motion as opposed to, say, the distance along the track. A very instructive exercise would be to derive the above equation from Newton's second law. Good luck with that.

Example: Plane pendulum

Here we revisit the familiar equation of motion of a pendulum moving in a plane. We model it as a particle constrained to move in a circle of radius l. We put our x-y coordinates in the plane of the circle with y being the vertical direction and the center of the circle being the origin. Because the mass must move on the circle, at each time t we have

$$x(t)^2 + y(t)^2 = l^2, \quad z(t) = 0.$$

and we can use an angle θ to characterize the configuration of the pendulum, as usual,

$$x = l\cos\theta, \quad y = l\sin\theta.$$

A curve in configuration space is then characterized by the graph $\theta = \theta(t)$ via

$$x(t) = l\cos\theta(t), \quad y(t) = l\sin\theta(t).$$

The motion of the pendulum is, evidently, completely characterized by the variable θ. We call θ the *generalized coordinate* for this system. It arises because the motion is constrained by $x(t)^2 + y(t)^2 = l^2$ and $z(t) = 0$. Even though the pendulum swings in 3-d space, its motion is characterized by a single variable. We say the pendulum has "one degree of freedom". Notice that the roller coaster example has a similar state of affairs (exercise).

We now can compute the kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\left[(-\dot{\theta}\sin\theta)^2 + (\dot{\theta}\cos\theta)^2\right] = \frac{1}{2}ml^2\dot{\theta}^2.$$

The potential energy can be taken to be

$$V = mgl\sin\theta,$$

where we've chosen our reference point to be where $\theta = 0$. Note that this is *not* the vertical configuration of the pendulum.

Next we compute

$$L(\theta, \dot{\theta}) = T - V = \frac{1}{2}m^2l^2\dot{\theta}^2 - mgl\sin\theta,$$

and

$$\frac{\partial L}{\partial \theta} = -mgl\cos\theta, \quad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}.$$

The EL equation of motion is then

$$ml^2\ddot{\theta} + mgl\cos\theta = 0.$$

Using our conservation of energy approach to solving this equation of motion we have

$$t - t_0 = \int_{\theta_0}^{\theta} dx\sqrt{\frac{2}{m(E - mgl\sin x)}}.$$

From this expression it follows that the motion of θ in time involves (the inverse of) elliptic integral functions. Not trivial, but completely tractable. According to our general analysis, the motion of the pendulum is periodic, but the period will depend upon the amplitude. Not so good for time-keeping! How does the familiar sinusoidal behavior, with amplitude-independent period arise? It arises by considering motion near stable equilibrium. Stable equilibrium is at $\theta = -\frac{\pi}{2}$. Let us expand the cosine around that point via Taylor series. With $u = \theta + \frac{\pi}{2}$ being the displacement from equilibrium we get

$$\cos(u - \frac{\pi}{2}) = \cos(-\frac{\pi}{2}) - \sin(-\frac{\pi}{2})u + \ldots = u + \ldots$$

Since

$$\ddot{\theta}(t) = \ddot{u}(t),$$

the equation of motion is, near stable equilibrium, approximated by

$$ml^2\ddot{u} + mglu = 0, \quad u << 1.$$

5

The solutions give the familiar sinusoidal – harmonic oscillator – motion of a pendulum near stable equilibrium:

$$u = A\cos\omega t + B\sin\omega t, \quad \omega = \sqrt{\frac{g}{l}}.$$

Example: The Atwood Machine

We compute the equation of motion of the Atwood machine. The simplest version of the Atwood machine consists of two masses, m_1 and m_2, suspended on either side of a massless, frictionless pulley of radius R by a massless rope of length l.

Picking an origin in space at the center of the pulley, we can characterize the configuration of the system by, say, the vertical position of m_1, which we specify via the vertical *distance s* from the origin. The location of the other mass is now determined; its vertical distance from the origin is $l - \pi R - s$. Evidently, if the velocity of m_1 is $\dot{s}$, then the velocity of m_2 is $-\dot{s}$. The kinetic energy of the system (the two masses) is then

$$T = \frac{1}{2}m_1\dot{s}^2 + \frac{1}{2}m_2(-\dot{s})^2 = \frac{1}{2}(m_1 + m_2)\dot{s}^2.$$

The potential energy is*

$$V = -m_1 g s - m_2 g(l - \pi R - s) = -(m_1 - m_2)gs + (const.).$$

We will suppress the additive constant since only the derivative of V will be needed to make the EL equation. You can think of this as just re-adjusting the reference point for potential energy.

We have

$$L(s, \dot{s}) = T - V = \frac{1}{2}(m_1 + m_2)\dot{s}^2 + (m_1 - m_2)gs,$$

so the equation of motion is

$$0 = \frac{\partial L}{\partial s} - \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = (m_1 - m_2)g - (m_1 + m_2)\ddot{s},$$

i.e.,

$$\ddot{s} = \frac{(m_1 - m_2)}{(m_1 + m_2)}g.$$

It's actually not too hard to get this same equation via Newton's second law – you should try it. In your homework you will make the problem more realistic by letting the pulley have a mass and the ability to rotate. For this more sophisticated system the Lagrangian formalism becomes in many ways simpler than the Newtonian approach.

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* Note the minus signs in the potential energy since *s increases* as height *decreases.*

Generalized coordinates

The roller coaster, the pendulum, and the Atwood machine are examples which make use of configuration variables (x, θ and s) which are not simply the Cartesian coordinates of a particle in Euclidean space. Such variables, which generally arise to take account of some constraint imposed on the configuration of the system, are called generalized coordinates. So long as the generalized coordinate arises from the solution of an algebraic (as opposed to differential) equation imposed upon the Cartesian coordinates of a particle, it can be shown to be perfectly valid as a variable for forming the EL equations. I state this non-trivial result without proof; it is not obvious. The proof can be made from a mathematical point of view involving the behavior of the EL equations under changes of coordinates. From a physical point of view – at least for Newtonian systems – the proof stems from the fact that the forces enforcing the constraints do no work. This freedom to use generalized coordinates is one of the great practical benefits of the Lagrangian formalism. In particular, note that in the Newtonian formulation of the problem of, say, the plane pendulum, the forces which maintain the constraint (*e.g.,* the tension in the pendulum arm) must be accounted for in order to analyze the motion. In the Lagrangian approach one accounts for these "forces of constraint" simply by a judicious choice of coordinates.

Generalization to more than one degree of freedom

Although many simple systems can be usefully modeled with a single configuration variable, as above, more realistic systems require more variables. The minimum number of variables needed to uniquely characterize the configuration of the system is called the number of *degrees of freedom* of the mechanical system* These variables can, again, be coordinates of the generalized type, arising from the solution of some algebraic constraint equations. Usually we denote such generalized coordinates by q^{α}, $\alpha = 1, 2, \ldots, n$, where n is the number of degrees of freedom of the system.

Some examples are as follows. A particle moving in three dimensions has three degrees of freedom and the generalized coordinates can be taken to simple be the Cartesian coordinate, $(q^1, q^2, q^3) = (x, y, z)$, relative to some choice of origin. One can also work with curvilinear coordinates. A less trivial example is a particle constrained to move on the surface of a sphere, *e.g.,* in a *spherical pendulum.* If the sphere has radius R, then we can take as our generalized coordinates the latitude and longitude, *i.e.,* the usual spherical polar coordinates $(q^1, q^2) = (\theta, \phi)$. These provide the solution to

$$x^2 + y^2 + z^2 = R^2,$$

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* This is *not* quite the same thing as the "degrees of freedom" used to characterize the thermal energy via the equipartition theorem.

via

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta.$$

With more degrees of freedom we can still build the Lagrangian as the difference between kinetic and potential energies $L = L(q, \dot{q}, t)$. Although I will not prove it, it is certainly plausible – and is true! – that one can then generalize our variational principle approach to equations of motion – known as *Hamilton's principle*, by the way – to systems with more than one degree of freedom. One simply computes one EL equation for each degree of freedom. The form of the EL equations is

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} = 0.$$

Let us look at the above two examples to illustrate this point.

Example: Newtonian particle in 3-d

In Cartesian coordinates we have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z, t)$$

and the EL equations are

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{\partial V}{\partial x} - m\ddot{x}.$$

$$0 = \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = -\frac{\partial V}{\partial y} - m\ddot{y}.$$

$$0 = \frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = -\frac{\partial V}{\partial z} - m\ddot{z}.$$

Here you can see we have just recovered the three Cartesian components of $\vec{F} = m\vec{a}$.

As already mentioned, one of the great strengths of the Lagrangian approach is that it facilitates the use of curvilinear coordinates. For example, in spherical polar coordinates (r, θ, ϕ) we have

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2).$$

Thus, expressing the potential in terms of (r, θ, ϕ) we have

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V.$$

It is now very straightforward to compute the equations of motion:

$$0 = \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 - m\ddot{r},$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = mr^2 \sin\theta\cos\theta\dot\phi^2 - \frac{\partial V}{\partial \theta} - m\frac{d}{dt}(r^2\dot\theta)$$

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = -\frac{\partial V}{\partial \phi} - m\frac{d}{dt}(r^2\sin^2\theta\dot\phi),$$

From these formulas we can easily see what to do for the spherical pendulum. Let the center of the pendulum be the origin of spherical polar coordinates, with gravity acting in the z direction. We set

$$r(t) = R, \quad V = mgR(1 + \cos\theta).$$

Here we have chosen the reference point of potential energy to be at the point of stable equilibrium, $\theta = \pi$. The equations of motion for (θ, ϕ) reduce to*

$$\ddot\theta - \frac{g}{R}\sin\theta - \sin\theta\cos\theta\dot\phi^2 = 0,$$

$$\frac{d}{dt}(\sin^2\theta\dot\phi) = 0.$$

The first equation shows the effects of gravity. Note in particular that if $\dot\phi = 0$ then the motion is just that of a planar pendulum. The second equation expresses conservation of the z component of angular momentum about the origin. Note that if we "turn off" gravity by setting $g = 0$, we are considering (otherwise) "free" motion on a sphere. In this context note that a solution to the equations of motion with $g = 0$ is simply $\theta(t) = \frac{\pi}{2}$ and $\phi = \omega t$. This is uniform motion on a great circle.

Symmetries and Conservation Laws

One of the most profound features of the Lagrangian formulation of mechanics is a deep way of thinking about the existence of conserved quantities. These ideas first took their complete and modern form with the work of Emmy Noether in 1915, although the ideas we shall discuss here go back much further than that. One could spend many lectures on this topic; we will only have time for a brief and superficial introduction.

Cyclic coordinates and conservation of canonical momentum

The bulk of the conservation laws which we use to analyze dynamical systems can be viewed as arising from a very simple observation about the EL equations, which for $L = L(q, \dot q, t)$ take the form

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\alpha} = 0, \quad \alpha = 1, 2, \dots.$$

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* You can easily see that the equations of motion for r are now trivial.

Suppose a particular coordinate, say, q^1, is missing from the Lagrangian; it simply doesn't appear for whatever reason. Then we have

$$\frac{\partial L}{\partial q^1} = 0.$$

This means that the differential equation of motion associated with the number 1 degree of freedom takes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^1}\right) = 0.$$

This motivates the definition of the *momentum canonically conjugate to q^α*, or simply *the canonical momentum*

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}.$$

Note that, like L in general p_α is some function of $(q, \dot{q}, t)$. The result we have now is if a variable q^α is absent from the Lagrangian – for historical reasons q^α is said to be "cyclic" – then the corresponding canonical momentum is conserved *when the equations of motion are satisfied*:

$$\frac{\partial L}{\partial q^\alpha} = 0 \quad \Longrightarrow \quad \frac{d}{dt}p_\alpha = 0.$$

Let's look at a couple of key examples.

First, consider a particle moving near the surface of the Earth in a uniform gravitational field. With the z axis oriented along the vertical direction, the Lagrangian takes the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz.$$

As you can see, both x and y are cyclic. Thus the corresponding momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

are conserved. This should make sense to you - there is no force in the x and y directions and so the corresponding momentum components are conserved. More generally, for any particle moving in 3-d with the standard Newtonian type of Lagrangian

$$L = T - V,$$

you can check that the three canonical momenta are just the usual mass times velocity, and a cyclic coordinate just means that the component of force in that direction vanishes. This latter observation explains why p_α are called "momenta".

As another kind of example, consider a particle moving in 3-d in a spherically symmetric, central force field. The potential is

$$U(x, y, z) = V(r),$$

10

and in spherical polar coordinates the Lagrangian takes the form

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r).$$

Notice that the variable ϕ is cyclic. The corresponding conserved momentum is

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\sin^2\theta\dot{\phi}.$$

The meaning of this conservation law is the z component of angular momentum. Indeed, you can check that if the motion happens to lie in the x-y plane ($\theta = \pi/2$), then the angular momentum takes a familiar form. Now, in this spherically symmetric problem any direction could be chosen to be the z direction; we conclude that *any* component of angular momentum will be conserved.

We have seen how conservation laws arise from the EL equations. What does this have to do with symmetry? Simple. If a coordinate is absent from the Lagrangian, then the Lagrangian doesn't change if we change that coordinate. The Lagrangian is invariant under changes of the cyclic coordinate – we say that it admits a symmetry.* The specific type of symmetry depends upon the meaning of the cyclic coordinate. If it is a Cartesian coordinate we speak of translational symmetry and conservation of linear momentum. If the cyclic coordinate is an angular variable, we speak of rotational symmetry and conservation of angular momentum.

What about the most famous conservation law: conservation of energy? It can be understood as coming from a symmetry, too, though it is slightly more work to see it. The symmetry corresponding to energy conservation is time translational symmetry. We say a Lagrangian has time translation symmetry if it is the same function of $(q, \dot{q})$ at each time t. This just means that L has no explicit time dependence, *i.e.*,

$$L = L(q, \dot{q}) \quad \Longleftrightarrow \quad \frac{\partial L}{\partial t} = 0.$$

Now, consider the following computation of the time rate of change of the Lagrangian along a curve $q^\alpha = q^\alpha(t)$:

$$\frac{d}{dt}L = \frac{\partial L}{\partial t} + \sum_\alpha \left(\frac{\partial L}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial L}{\partial\dot{q}^\alpha}\ddot{q}^\alpha\right).$$

Suppose the curve is not just any old curve, but satisfies the EL equations of motion. Then we can replace

$$\frac{\partial L}{\partial q^\alpha} = \frac{d}{dt}\frac{\partial L}{\partial\dot{q}^\alpha},$$

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* What does *symmetry* mean, anyway? One definition is "change without change". You change one thing, and some other thing doesn't change, *i.e.,* can't tell the first change was made. In our case, you change a cyclic coordinate value and the Lagrangian can't tell.

to get

$$\frac{d}{dt}L = \frac{\partial L}{\partial t} + \sum_\alpha \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\ddot{q}^\alpha \right).$$

Moving the "total time derivative" to the right, and the partial time derivative to the left, then using the product rule for derivatives, we get

$$\frac{d}{dt}(p_\alpha \dot{q}^\alpha - L) = -\frac{\partial L}{\partial t}.$$

This relationship holds on any curve satisfying the Euler-Lagrange equations of motion. We define the *canonical energy* $E = E(q, \dot{q}, t)$ to be

$$E = p_\alpha \dot{q}^\alpha - L.$$

We say that if the Lagrangian is time translation symmetric, *i.e.*, $\frac{\partial L}{\partial t} = 0$, then the energy is conserved when the equations of motion hold:

$$\frac{dE}{dt} = 0.$$

Why do we call E the energy? Just look at a Newtonian particle with a time translation invariant potential energy function, we have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(\vec{r}),$$

and

$$(p_1, p_2, p_3) = (m\dot{x}, m\dot{y}, m\dot{z}),$$

and

$$E = (m\dot{x})\dot{x} + (m\dot{y})\dot{y} + (m\dot{z})\dot{z} - L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(\vec{r}) = T + V.$$

Let us consider a simple example. Recall the plane pendulum. Let ϕ denote the angle measured from the equilibrium position of the pendulum. The Lagrangian is:

$$L(\phi, \dot{\phi}) = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi).$$

Suppose we have a means of changing the length of the pendulum. For example, maybe the pendulum is constructed by a string passing through a hole in a table connected to a mass, and we can draw the string into or out of the hole. To be concrete, suppose we pull the string in so that its length is a function of t of the form

$$l(t) = l_0 - bt,$$

where b is a constant. We suppose that b is small enough such that we can still model the pendulum at any given time as still being characterized by the angular variable ϕ and the angular velocity $\dot{\phi}$.† The Lagrangian is now an explicit function of t:

$$L(\phi, \dot{\phi}, t) = \frac{1}{2}ml(t)^2\dot{\phi}^2 - mgl(t)(1-\cos\phi) = L(\phi, \dot{\phi}) = \frac{1}{2}m(l_0-bt)^2\dot{\phi}^2 - mg(l_0-bt)(1-\cos\phi).$$

The equations of motion are

$$\frac{d}{dt}\left(l(t)^2\frac{d\phi(t)}{dt}\right) + gl(t)\sin(\phi(t)) = 0.$$

The energy is given by

$$\begin{aligned} E &= \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L \\ &= \frac{1}{2}ml(t)^2\dot{\phi}^2 + mgl(t)(1 - \cos\phi) \\ &= \frac{1}{2}m(l_0 - bt)^2\dot{\phi}^2 + mg(l_0 - bt)(1 - \cos\phi). \end{aligned}$$

The time rate of change of energy is given by

$$\begin{aligned} \frac{dE}{dt} &= ml(t)\frac{dl(t)}{dt}\dot{\phi}^2 + mg\frac{dl(t)}{dt}(1 - \cos\phi) + \dot{\phi}\left(ml(t)^2\ddot{\phi} + mgl(t)\cos\phi\right) \\ &= m\frac{dl(t)}{dt}\left(l(t)\dot{\phi}^2 + mg(1 - \cos\phi)\right) \end{aligned}$$

The second equality arises by assuming $\phi(t)$ satisfies the equation of motion. You can easily see that the energy is conserved if and only if $l(t)$ is constant in time (*i.e.*, $b = 0$ in our example), which is equivalent to demanding that $\frac{\partial L}{\partial t} = 0$.

To summarize, a symmetry is a transformation of the system which does not change the Lagrangian. Associated to every symmetry is a conservation law. As it turns out, all known conservation laws follow this pattern! In particular, for a closed system modeled by Newtonian mechanics, the homogeneity of space and time and the isotropy of space* always yield a Lagrangian which has 3-d spatial translational symmetry, 3-d rotational symmetry, and time translational symmetry. This reflects the fact that the *closed system* behaves the same no matter where it is in space, no matter its orientation in space, and no matter when one considers the system. Thus closed (Newtonian) systems always have conservation of total energy, linear momentum, and angular momentum. We shall see an example of this when we study the "2-body problem".

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† We say that l is being changed "adiabatically".
* These properties of space-time are tacitly assumed in Newtonian mechanics; they became substantially modified when relativistic gravitational effects are taken into account.

A variational principle for a Newtonian particle in one dimension

Before getting into the generalities, let us get a feel for what is going on with a simple example. Consider a particle moving in one dimension (or some other system with a one-dimensional configuration space); moving under the influence of a possibly time-dependent force. We parametrize the configuration space with $x \in R^1$, and the force is $\vec{F} = f(x,t)\hat{i}$. Note that, for simplicity, we defer consideration of velocity dependent forces. The equation of motion *a la* Newton is then

$$m\frac{d^2x(t)}{dt^2} = f(x(t),t).$$

We note that all one-dimensional forces admit a potential energy function $V(x,t)$ such that

$$f(x,t) = -\frac{\partial V(x,t)}{\partial x}.$$

(As an exercise you should prove this!) So the equation of motion can be written as

$$m\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} = 0.$$

From Newton's point of view, this equation of motion arises from a postulate – his second law. We now will see how to obtain this equation of motion from a very different kind of reasoning involving a *variational principle*.

The idea of a variational principle is really not that difficult to grasp, but it *is* a little different from what you are used to, I expect. In qualitative terms, the variational principle considers all possible paths the particle can take and assigns a measure of "goodness" or "fitness" to each path. By optimizing the goodness one selects a privileged path or, more commonly, privileged paths. Let us make this mathematically precise.

We begin by considering paths $x(t)$, which range between fixed initial and final points, x_1 at $t = t_1$ and x_2 at $t = t_2$, that is,

$$x(t_1) = x_1, \quad x(t_2) = x_2.$$

As a random, simple example,

$$x(t) = \frac{t - t_2}{t_1 - t_2}x_1 + \frac{t - t_1}{t_2 - t_1}x_2,$$

and

$$x(t) = x_1 \sin\left(\frac{\pi}{2}\frac{t - t_2}{t_1 - t_2}\right) + x_2 \sin\left(\frac{\pi}{2}\frac{t - t_1}{t_2 - t_1}\right),$$

are such paths. There are, of course, infinitely many paths connecting any given endpoints. Note that these paths will not, in general, satisfy Newton's second law for the specified force.

14

Next we define a *functional* $S[x]$ on the set of paths described above. $S[x]$ is called the *action* functional; the action is a rule that associates a number to each path satisfying the given boundary conditions. For the Newtonian particle of mass m moving in the force described by the potential energy V we define $S[x]$ by

$$S[x] = \int_{t_1}^{t_2} dt \, \left(\frac{1}{2}m(\frac{dx(t)}{dt})^2 - V(x(t), t) \right) .$$

You will recognize the integrand, called the *Lagrangian,** as the difference of kinetic $T(t)$ and potential $V(t)$ energies along the curve $x(t)$,

$$S[x] = \int_{t_1}^{t_2} dt \, L(t) \, ,$$

$$L(t) = T(t) - V(t).$$

Given a curve, $x(t)$, it is easy to see how to compute the number assigned by the action functional to $x(t)$ from the formula above: just compute $L(t)$ for that curve and integrate.

As an example, suppose $V(x, t) = mgx$, *i.e.,* we have a particle moving in a uniform gravitational field. Let us evaluate the action for the path

$$x(t) = \frac{t - t_2}{t_1 - t_2}x_1 + \frac{t - t_1}{t_2 - t_1}x_2.$$

(Note that this path is *not* a solution to the equation of motion – it wouldn't occur in classical mechanics as an actual motion of the system.) We get

$$S = \frac{m}{t_2 - t_1} \left\{ \frac{1}{2}(x_2 - x_1)^2 - \frac{1}{2}g(t_2 - t_1)^2(x_2 + x_1) \right\} .$$

Verifying this result is a good exercise for you — try it!

We now consider the problem of finding *critical points* of the action functional $S[x]$. The critical points will be the "best" paths we are seeking. Recall from elementary calculus that the critical points x_0 of a function $f(x)$ are points where the derivative of f vanishes: $f'(x_0) = 0$. What this means is that a small displacement from x_0 does not change the value of the function to first-order in the displacement from that point. To see this, just write out the Taylor series. If x_0 is a critical point for the function f, and setting $\epsilon = x - x_0$, we have

$$f(x_0 + \epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{1}{2}f''(x_0)\epsilon^2 + \ldots = f(x_0) + \text{terms of order } \epsilon^2.$$

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* As we shall see, strictly speaking the Lagrangian should be viewed as a function on the extended velocity phase space. The integrand of the action integral is actually the Lagrangian evaluated on a curve.

Likewise we say that *a curve $x(t)$ is a critical point of $S[x]$ if a small change in the function does not alter the value of $S[x]$ to first order in the change in the function.* So, if a curve $x(t)$ is a critical point of the action, then if we change the curve, say, to $x(t) + \delta x(t)$, where $\delta x(t)$ is an arbitrary function (except for boundary conditions – see below) the action should be unchanged to first order in δx.

Note: The function $\delta x(t)$ is called the *variation* of $x(t)$.

Recall we are considering paths that begin and end at some fixed points (x_1 and x_2). And since $x(t)$ is already assumed to have those endpoints, in order for the varied path $x(t) + \delta x(t)$ to have the correct boundary conditions the variation must satisfy

$$\delta x(t_1) = \delta x(t_2) = 0.$$

Let us now compute the change in action to first order in the variation. Computing to first order in the variation we get (try it!)

$$S[x + \delta x(t)] = \int_{t_1}^{t_2} dt \, \left(\frac{1}{2} m \left(\frac{dx(t)}{dt} + \frac{d\delta x(t)}{dt} \right)^2 - V(x(t) + \delta x(t), t) \right)$$

$$= \int_{t_1}^{t_2} dt \, \left(\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + m \frac{dx}{dt} \frac{d\delta x}{dt} + \frac{1}{2} m \left(\frac{d\delta x}{dt} \right)^2 - V(x, t) - \frac{\partial V}{\partial x} \delta x + \ldots \right)$$

$$= S[x] + \int_{t_1}^{t_2} dt \, \left(m \frac{dx(t)}{dt} \frac{d\delta x(t)}{dt} - \frac{\partial V(x, t)}{\partial x} \bigg|_{x=x(t)} \delta x(t) \right) + \mathcal{O}(\delta x^2).$$

The second equality came by expanding out the square in the kinetic energy and by using Taylor series to expand $V(x + \delta x)$ to first order in δx. The strategy is now to see what conditions the curve $x(t)$ must satisfy so that the $\mathcal{O}(\delta x)$ term vanishes for any choice of $\delta x(t)$.† To this end, we integrate by parts in the first term of that integral; the endpoint terms do not contribute because δx vanishes at the endpoints:

$$\int_{t_1}^{t_2} dt \, m \frac{dx(t)}{dt} \frac{d\delta x(t)}{dt} = m \frac{dx(t_2)}{dt} \delta x(t_2) - m \frac{dx(t_1)}{dt} \delta x(t_1) - \int_{t_1}^{t_2} dt \, m \frac{d^2 x(t)}{dt^2} \delta x(t)$$

$$= - \int_{t_1}^{t_2} dt \, m \frac{d^2 x(t)}{dt^2} \delta x(t)$$

So we get for our critical point condition (good exercise)

$$- \int_{t_1}^{t_2} \left(m \frac{d^2 x(t)}{dt^2} + \frac{\partial V(x, t)}{\partial x} \bigg|_{x=x(t)} \right) \delta x(t) \, dt = 0$$

† The order δx term is called the *first variation* of the action and is usually denoted by δS. The critical point condition is thus expressed as $\delta S = 0$.

Since this must hold for any function $\delta x(t)$ in the interval $t_1 < t < t_2$ (subject to its vanishing at the endpoints), it follows that the critical point $x(t)$ must satisfy Newton's second law:

$$m\frac{d^2x}{dt^2} + \frac{\partial V(x,t)}{\partial x}\bigg|_{x=x(t)} = 0, \quad t_1 < t < t_2.$$

This can be made quite rigorous given appropriate statements about the smoothness of the functions being used. The idea of the proof is that we can choose $\delta x(t)$ to be arbitrarily well localized about any point t in the interval $t_1 < t < t_2$, and this forces the rest of the integrand to vanish in an arbitrarily small neighborhood of that point. Continuity does the rest.

To summarize: Newton's second law can be viewed as arising from a *variational principle:** Physical trajectories $x(t)$ (obeying the second law) are critical points of the functional $S[x] = \int dt\, L$, where $L = T - V$. This characterization of a dynamical system is quite general and is useful for a wide variety of physical systems — not just mechanical systems, either.

A small technical digression

It is often asserted that the action is minimized by a curve satisfying the equations of motion, but this is by no means necessary. As in ordinary calculus, the existence of a critical point signals the existence of either a local maximum/minimum or a saddle point. We can investigate this a bit further and show that if the time interval $T = t_2 - t_1$ is sufficiently short the action analyzed above is minimized, however. Let's briefly see how this goes. This is a little bit technical for our course, so skip it if you wish.

For later simplicity, we set $t_1 = 0$ and $t_2 = T$. We can decide on the nature of the critical point by expanding the action to *second* order in the variations. Granted that $x(t)$ is a critical point, we have

$$S[x + \delta x] = S[x] + \delta^2 S + \mathcal{O}(\delta x^3),$$

where $\delta^2 S$ is called the *second variation* of the action about the critical point. A simple computation shows that, for the 1-d Newtonian system we have that (exercise)

$$\delta^2 S = \int_0^T dt\, \frac{1}{2}\left[m(\delta\dot{x})^2 - g(t)\delta x^2\right],$$

where

$$g(t) = \frac{\partial^2 V(x,t)}{\partial x^2}\bigg|_{x=x(t)}.$$

_____

* The term "variational principle" arises because we consider the change in the functional as we *vary* the possible paths in the vicinity of a critical point.

Our goal is to see if the second variation is positive, negative, or zero — corresponding to $x(t)$ being a local minimum, maximum, saddle point, respectively. To this end we assume that $g(t)$ is a continuous function of t; we then have the simple estimate

$$- \int_0^T dt \, \frac{1}{2} f(t) \delta x^2 \geq -C \int_0^T dt \, \frac{1}{2} \delta x^2,$$

where the constant C is given by

$$C = \sup_t (f(t)).$$

Thus we have

$$\delta^2 S \geq \int_0^T dt \, \frac{1}{2} \left[m(\delta \dot{x})^2 - C \delta x^2 \right],$$

I think you can see that the kinetic energy term provides a positive contribution, while the potential energy term provides a negative contribution. Thus, in general, one cannot assert that a minimum occurs. Still, we can say a bit more. Recall that δx is a function on the interval $[0, T]$ which vanishes at the end points. We can express it as a Fourier series:

$$\delta x = \sum_{n=1}^{\infty} a_n \sin(\frac{n \pi t}{T}).$$

This gives (exercise)

$$\delta^2 S \geq \frac{T}{4} \sum_{n=1}^{\infty} \left[m(\frac{n \pi}{T})^2 - C \right] a_n^2.$$

For a given potential energy function C is a fixed constant. You can see from the above expression that, given C, we can always pick T small enough such that the first term in square brackets dominates the second. Thus for T sufficiently small we have that the second variation is positive and $x(t)$ defines a local minimum of the action functional.

The Euler-Lagrange equations

We have seen that the curves which are critical points of the action, constructed as the integral of $T - V$, are precisely the curves obeying Newton's second law. Now I would like to examine this result from a more general perspective.

To begin, notice that the Lagrangian is a formula for constructing from any given curve $x = x(t)$ a function of time which can be integrated. This formula involves the curve itself, $x(t)$, and its first derivative, $\frac{dx}{dt}$. The formula may also depend upon t if V depends explicitly on time. Let us view the Lagrangian, then, as a function of 3 variables which we

denote as x, $\dot{x}$, and t. We could use any labels we want, of course, but these labels help us remember what we are doing. We have

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - V(x, t).$$

It is important to be clear that I am not, at this point, viewing $\dot{x}$ as a derivative; it's just another variable. The derivative comes in when we build the action functional. Given a curve,

$$x = h(t),$$

the action is constructed as

$$S[h] = \int_{t_1}^{t_2} dt\, L(h(t), \frac{dh(t)}{dt}, t).$$

To make things maximally clear I have used $h(t)$ instead of the more traditional $x(t)$ to denote the curve. This is so you can see that the $L(x, \dot{x}, t)$ is a function of 3 variables and the integrand of the action integral,

$$L(t) = L(h(t), \frac{dh(t)}{dt}, t),$$

is a function of one variable. You may think I am getting overly pedantic here, but this failure to distinguish x from $h(t)$ and $\dot{x}$ from $\frac{dh(t)}{dt}$ probably causes more confusion in this business than anything else.

Now I would like to repeat the analysis of the critical points using this more general notation. Suppose, then, $h(t)$ is such that the curve $x = h(t)$ is a critical point of the action. Then if we vary the curve, $x = h(t) + \delta x(t)$, where $\delta x(t)$ is any function satisfying

$$\delta x(t_1) = \delta x(t_2) = 0,$$

then the action should not change to first order in the variation $\delta x(t)$. We have

$$S[h + \delta x] = \int_{t_1}^{t_2} dt\, L(h(t) + \delta x(t), \frac{dh(t)}{dt} + \frac{d\delta x(t)}{dt}, t)$$

Now we want to use the multi-variable version of the Taylor expansion of a function – to first order, anyway. For any function $F(x, y)$ and a given point (x_0, y_0) we have

$$F(x, y) = F(x_0, y_0) + \frac{\partial F}{\partial x}\Big|_{x_0, y_0}(x - x_0) + \frac{\partial F}{\partial y}\Big|_{x_0, y_0}(y - y_0) + \dots.$$

Here the dots mean terms of quadratic or higher order in the displacement from (x_0, y_0). At each t, the Lagrangian is a function of two variables (its first two arguments) and we

19

clearly are displacing from the values $x = h(t)$ and $\dot{x} = \frac{dh(t)}{dt}$. Using our Taylor expansion result we have

$$L(f(t)+\delta h(t), \frac{dh(t)}{dt}+\frac{d\delta h(t)}{dt}, t) = L\Big|_{x=h,\dot{x}=\frac{dh}{dt}} + \frac{\partial L}{\partial x}\Big|_{x=h,\dot{x}=\frac{dh}{dt}}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\Big|_{x=h,\dot{x}=\frac{dh}{dt}}\frac{d\delta x(t)}{dt} + \dots$$

So, to first order in the displacement from the would-be critical point we have

$$S[h + \delta x] = S[h] + \delta S + \dots,$$

where

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\frac{d\delta x}{dt}\right).$$

The quantity δS is called *the first variation* of the action.* Let us check this general formula for our example treated earlier. We have

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - V(x, t),$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = f,$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p,$$

so that

$$\delta S = \int_{t_1}^{t_2} dt \left(f\delta x + p\frac{d\delta x}{dt}\right),$$

where it is understood that the integrand is computed via $x = h(t)$ and $\dot{x} = \frac{dh}{dt}$. If you go back and look at our initial computation of the critical point condition you will see that we computed precisely this δS.

The next step, you will recall, is to demand that the first variation of the action vanishes for any choice of the function $\delta x(t)$. To this end we have to integrate by parts in the second term. We write

$$\frac{\partial L}{\partial \dot{x}}\frac{d\delta x}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \delta x\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}.$$

Substituting this into the first variation, we see that we can explicitly integrate the d/dt term to get

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x + \left[\frac{\partial L}{\partial \dot{x}}\delta x\right]_{t_1}^{t_2}.$$

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* Notice that the first variation is computed by taking partial derivatives of the Lagrangian with respect to the variables x and $\dot{x}$. At this step these quantities are variables, not functions of t! You can see why it is important to view $\dot{x}$ as a variable, not the derivative of a function of t. After all, there is no definition of the derivative of a function with respect to another function in calculus!

Again, **it is understood that everything is evaluated on the curve** $x = h(t)$, *e.g.*,

$$\frac{\partial L}{\partial \dot{x}} \implies m\frac{dh(t)}{dt}$$

in the above integral.

Because the variation of the curve vanishes at the endpoints, the last term vanishes. Because the first variation of the action must vanish for any choice of $\delta x(t)$, the integrand of the first term must vanish. We get the following differential equationfor $h(t)$, known as the Euler-Lagrange equation:

$$\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]_{x=h(t),\dot{x}=\frac{dh(t)}{dt}} = 0.$$

We usually just write this as

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0,$$

but it is important to keep in mind that this is not some kind of equation for L. The Euler-Lagrange equation is a formula for creating a differential equation of motion in x.

Let's check that this works for

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - V(x,t).$$

We have (tacitly assuming everything gets evaluated on the curve):

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = f(h(t), t),$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \implies \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\frac{d^2h(t)}{dt^2},$$

so that the Euler-Lagrange equation is

$$f(h(t), t) - m\frac{d^2h(t)}{dt^2} = 0,$$

which is precisely Newton's second law for the curve $x = h(t)$.

I have been pretty pedantic with the notation to try and mitigate the inevitable misconceptions about how to use the formulas. Let me show you the more usual notation we use in this Lagrangian game. As usual, we view the Lagrangian as $L = L(x, \dot{x}, t)$, *e.g.*,

$$L = \frac{1}{2}m\dot{x}^2 - V(x,t).$$

With the notational understanding that

$$\frac{d}{dt}x = \dot{x}, \quad \frac{d}{dt}\dot{x} = \ddot{x},$$

21

the Euler-Lagrange equation for a Newtonian particle moving in 1-d in a potential $V(x, t)$ is usually written as

$$m\ddot{x} + \frac{\partial V}{\partial x} = 0.$$

Variational Principles in General

It is not too hard to generalize the foregoing in two ways: (1) more degrees of freedom, (2) an arbitrary Lagrangian. I won't trouble you with the details of the proofs, but simply show you the result.

Let the system of interest be characterized by n generalized coordinates q^α, $\alpha = 1, 2, \ldots, n$. Consider an action $S[q]$, which is a functional of "curves" $q^\alpha = q^\alpha(t)$ between fixed endpoints, where t is some parameter. We assume the action is the integral of a Lagrangian,

$$L = L(q, \dot{q}, t),$$

$$S = \int_{t_1}^{t_2} dt\, L(q(t), \frac{dq(t)}{dt}, t),$$

as discussed previously. The curve $q^\alpha = q^\alpha(t)$ is a critical point of the action $S[q]$ if and only if the curve satisfies the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} = 0, \quad \alpha = 1, 2, \ldots, n.$$

Example: Geodesics in the Euclidean Plane

Everyone has heard: "The shortest distance between two points is a straight line." This is not always true – think of motion on the surface of the Earth! But for motion in Euclidean space this adage – characterizing the *geodesics* of Euclidean space – is true. Let us prove it.

Let (x_1, y_1) and (x_2, y_2) be two points in the plane. Let $(x, y) = (x(t), x(t))$ be a curve connecting these points*, so that

$$x(t_1) = x_1, \quad y(t_1) = y_1,$$

$$x(t_2) = x_2, \quad y(t_2) = y_2.$$

We want to study the length of this curve and adjust the curve to make the length as short as possible. The (infinitesimal) distance between a point (x, y) and the nearby point $(x + dx, y + dy)$ is given by

$$ds = \sqrt{(dx^2 + dy^2)}.$$

* Warning: Notational abuse in what follows.

Therefore the length of the curve is

$$S = \int_{t_1}^{t_2} dt \left[\left(\frac{dx(t)}{dt} \right)^2 + \left(\frac{dy(t)}{dt} \right)^2 \right]^{1/2}.$$

Evidently S is a functional of the curve with a Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

We want to find the curve with the smallest length; this will be a critical point of S and so must satisfy the Euler-Lagrange equations. We have

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot{x}} = \left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right),$$

and

$$\frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial \dot{y}} = \left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right).$$

The Euler-Lagrange equations are:

$$\frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0, \quad \frac{d}{dt} \left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0.$$

These equations imply there are constants c_1 and c_2 such that

$$\dot{x} = c_1 \sqrt{\dot{x}^2 + \dot{y}^2}, \quad \dot{y} = c_1 \sqrt{\dot{x}^2 + \dot{y}^2}.$$

Evidently

$$\dot{x} = (const.)\dot{y},$$

so that for displacements along the curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = constant,$$

which means the curve is a straight line.