Physics 3550

Newton's Third Law. Multi-particle systems.

Relevant Sections in Text: §1.5, 3.1, 3.2, 3.3

Newton's Third Law.

You've all heard this one.

Actioni contrariam semper et qualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse quales et in partes contrarias dirigi.

To every action there is always an equal and opposite reaction: or the forces of two bodies on each other are always equal and are directed in opposite directions.

The idea here is the following. Often we consider a single body (or particle) with some forces on it. That's fine; it's an example of an *open system.* The body interacts with its environment. Newton's third law proposes that all forces stem from interactions between bodies and that these forces have a very symmetric character. In particular, if particle 1 exerts a force $\vec{F}_{12}$ on particle 2, and if particle 2 exerts a force $\vec{F}_{21}$ on particle 1, then these forces are related by

$$\vec{F}_{12} = -\vec{F}_{21}.$$

A good example of a force obeying the third law is the Newtonian gravitational force* between two point masses m_1 and m_2. With G being Newton's constant, we have

$$F_{ij} = \frac{Gm_i m_j}{|\vec{r}_i - \vec{r}_j|^3}\,(\vec{r}_i - \vec{r}_j).$$

You can easily see that Newton's third law is satisfied here. You can write down a mathematically identical version of the law of electrostatic attraction and repulsion, thus seeing that it satisfies Newton's third law.

Newton's third law, like any other law, is subject to experimental verification. It holds in many situations. However, it is not hard to come up with a situation where the third law is violated. Consider two electric charges, both moving toward a fixed point of space with constant orthogonal velocities. Consider the instant of time when one of the charges – charge 1, say –is at that fixed point of space (and still moving), directly ahead of the other charge – charge 2. The net force between the charges does not obey Newton's third law (the electric force does, it is the magnetic force which causes this problem for Newton). In particular, the magnetic force of charge 2 on charge 1 is zero because its relative position

* I use the qualifier "Newtonian" here to emphasize that this force law is due to Newton and that it has been superseded by Einstein's theory of gravity, contained in his general theory of relativity.

vector is parallel to charge 2's velocity. The magnetic force of charge 1 on charge 2 is non-vanishing. This result is not a relativistic effect, *i.e.,* it occurs for arbitrarily slow speeds. Of course, Newton could not be aware of this feature of the magnetic force.

The principle consequence of Newton's third law (when it holds!) is conservation of momentum for a closed system. In this context, a "closed system" means that all forces have been accounted for in the definition of "the system". Let us first consider the total momentum of two bodies obeying Newton's third law. The total momentum $\vec{P}$ is defined as the vector sum of the individual momenta.* We model the bodies as particles, 1 and 2. We have

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2.$$

As the particles move their position vectors change in time, as we have discussed. In general, then, their velocities change in time yielding, in general, a time dependence for $\vec{P}$. The time rate of change is

$$\frac{d\vec{P}}{dt} = m_1\vec{a}_1 + m_2\vec{a}_2.$$

Using Newton's second law we have

$$\frac{d\vec{P}}{dt} = \vec{F}_1 + \vec{F}_2 = \vec{F}_{21} + \vec{F}_{12},$$

where the forces are the respective forces on each particle due to the other particle. These are the only forces by the "closed system" assumption. Using Newton's third law we have

$$\frac{d\vec{P}}{dt} = 0.$$

So neither particle's momentum will be conserved, but the total momentum will be conserved. Think of a binary star system.

It is not too hard to generalize this to any number of particles, if you don't mind a little notation. Suppose we have N particles, labeled by Latin letters, *e.g.,* $i, j = 1, 2, \ldots, N$. Let $\vec{F}_{ij}$ be the force of the i^{th} particle on the j^{th} particle, and suppose that these are *all* the forces on the system. Newton's third law tells us that

$$\vec{F}_{ij} = -\vec{F}_{ji}.$$

Newton's second law for the j^{th} particle due to the force of the i^{th} particle is

$$m_j\vec{a}_j = \vec{F}_{ij}.$$

* Why is momentum mass times velocity? Is it always defined that way? Why is the total momentum conserved? All these questions will be answered when we explore the Lagrangian form of mechanics.

Newton's second law for the j^{th} particle due to all the other particles is (using the superposition principle)

$$m_j \vec{a}_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} F_{ij}.$$

The total momentum is

$$\vec{P} = \sum_{i=1}^{N} m_i \vec{v}_i.$$

The time rate of change of the total momentum is, as before,

$$\frac{d\vec{P}}{dt} = \sum_{j=1}^{N} m_j \vec{a}_j = \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} F_{ij}.$$

Because of Newton's third law, every term in the double sum has a partner with which it cancels – can you see that? We have

$$\sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} F_{ij} = \vec{F}_{12} + \vec{F}_{13} + \ldots \vec{F}_{21} + \ldots + \vec{F}_{31} + \ldots = 0.$$

The particles whiz around with various time dependent positions, velocities and even accelerations, but the total momentum does not change in time.

The above computation shows that Newton's third law implies conservation of momentum for a closed system. It also shows that if all interparticle forces in a system obey the third law, the net force internal to the system is zero.

Suppose now that the system us subject to some *external* (not interparticle) forces, Let $\vec{F}_{ext}$ be the net external force on the system. From the discussion above, this is in fact the total net force on the system provided the inter-particle forces act pairwise and obey Newton's third law. Then we have the very important result:

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

It is this result that makes the particle model of an extended rigid body work so well. The body is made of a large number of particles, and it is held together by a large number of forces. But just so long as those forces obey Newton's third law, they all cancel when computing the time rate of change of the total momentum of the body. The body accelerates according to the net *external* force on the body. Let's make this more precise and more useful.

The total momentum is

$$\vec{P} = \sum_i m_i \vec{v}_i = \frac{d}{dt} \sum_i m_i \vec{r}_i.$$

We define the *center of mass* of the body by

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i,$$

where $M = \sum_i m_i$ is the total mass of the body, so that

$$\vec{P} = M \frac{d\vec{R}}{dt}.$$

The particle each move through space via $\vec{r}_i = \vec{r}_i(t)$, yielding $\vec{R} = \vec{R}(t)$. We can easily compute the acceleration of $\vec{R}$. We have, from our previous calculation,

$$M \frac{d^2\vec{R}}{dt^2} = \vec{F}_{ext}.$$

If we model the body as a particle with mass given by the total mass of the body and located at the center of mass, then – thanks to the third law – it obeys Newton's second law.

As a simple example of all this, consider a mechanical system consisting of two masses m_1 and m_2 connected by a spring, with spring constant k and equilibrium length L, and moving in a uniform gravitational field. * The force of mass 1 on mass 2 is

$$\vec{F}_{12} = -k(|\vec{r}_1 - \vec{r}_2| - L) \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_1 - \vec{r}_2|}.$$

Similarly, the force of mass 2 on mass 1 is given by

$$\vec{F}_{21} = -k(|\vec{r}_1 - \vec{r}_2| - L) \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}.$$

The gravitational force on masses 1 and 2 are given by

$$\vec{F}_1 = m_1 g \hat{k}, \quad \vec{F}_2 = m_2 g \hat{k},$$

where I've chosen the z direction to be the vertical direction. You can easily see that the net force on the *system of two masses* is just

$$\vec{F}_{net} = \vec{F}_{ext} = (m_1 + m_2) g \hat{k}.$$

* This is a simple model of a diatomic molecule near the surface of the earth!

One last point: Newton's third law for a closed system implies conservation of (total) momentum. We have seen that the magnetic force between two particles need not obey Newton's third law. What becomes of momentum conservation? The answer is that the momentum ledger must include contributions from the electric and magnetic fields. Do you recall that these fields carry energy and momentum? Taking this new electromagnetic "stuff" into account, momentum is in fact conserved.

Example: The Rocket

Rocket propulsion is one of the more fundamental and important examples of Newton's laws, particularly the third law. It is also slightly tricky, so it is worth explaining here. Recall that a rocket is a device for creating motion (acceleration, really). The rocket strategy is to keep throwing things out one side of the object you want to move. Conservation of momentum does the rest.

We work in the rest frame of the Earth and consider motion only in one dimension, which we call z. At time t let the rocket have mass $M(t)$ and velocity $\vec{V}(t) = V(t)\hat{k}$. For now we assume that there are no external forces on the rocket – the rocket *and its fuel* is a closed system. Thus the momentum of the system is conserved; we shall use this in a moment. Let a small time dt pass. Some fuel is burned and exhaust gases expelled. We have

$$M(t+dt) = M(t) + dM, \quad V(t+dt) = V(t) + dV.$$

(Here the differentials should be thought of as sufficiently small physical quantities. Of course, we will take the limit as they become vanishingly small when it is convenient.) Evidently, $dM < 0$. Let v_0 be the speed at which the mass is expelled *relative to the rocket*; we assume this is constant in time and is fixed by the rocket's design. The velocity at which mass is expelled relative to the Earth is $(V(t) - v_0)\hat{k}$. It is this velocity that will feature in conservation of momentum as computed in the inertial reference frame* fixed relative to the Earth. Conservation of momentum tells us that the momentum at time $t + dt$ is the same as the momentum at time t. The total momentum at time t is just $M(t)V(t)\hat{k}$. The rocket's momentum at time $t + dt$ is

$$(M(t) + dM)(V(t) + dV)\hat{k}$$

The expelled fuel's momentum at time $t + dt$ is

$$(-dM)(V(t) - v_0)\hat{k}.$$

So, conservation of *total* momentum says:

$$(M(t) + dM)(V(t) + dV) - dM(V(t) - v_0) = M(t)V(t).$$

* Newton's first law!

Since the differentials are (arbitrarily) small quantities, we can neglect their products relative to the linear terms. Thus we have

$$dM\,V + MdV - dM(V - v_0) = 0.$$

This simplifies to

$$MdV = -v_0 dM.$$

Dividing both sides by dt and taking the limit as the infinitesimals become arbitrarily small we get

$$M(t)\frac{dV(t)}{dt} = -v_0\frac{dM(t)}{dt}.$$

A very simple equation controlling the acceleration of the rocket!

We can integrate this equation easily enough. We write it as

$$\frac{dV}{dt} = -v_0\frac{d\ln M}{dt},$$

This has a solution

$$V(t) = v_0 \ln\left(\frac{M_0}{M(t)}\right) + V_0.$$

Here M_0 is the initial mass of the rocket, which it has when its speed is V_0.

It is easy to include external forces in this analysis. Let us add in gravity, with force $-M(t)g\hat{k}$. Now the change of momentum equation is

$$dMV + MdV - dM(V - v_0) = -Mgdt,$$

or

$$MdV = -v_0 dM - Mgdt \qquad (\star)$$

or

$$M(t)\frac{dV(t)}{dt} = -v_0\frac{dM(t)}{dt} - M(t)g.$$

Integrating equation $(\star)$ We get

$$V(t) = V_0 + v_0 \ln\left(\frac{M_0}{M(t)}\right) - g(t - t_0).$$

This formula gives the speed of the rocket in terms of the rocket's design parameters, namely v_0 and $M(t)$.

This formula explains why NASA's budget is so large — you only get a logarithmic bang for your fuel buck! Suppose the exhaust gases are moving at 2 km/s relative to the rocket, which Wikipedia says is a reasonable value. Suppose the rocket starts from rest. Let's see how much mass gets used up to get the rocket to Earth's escape velocity, which is 11.2 km/s. We assume that the rocket burns fuel such that that the burn time is, say, 100s. We have

$$v_0 = 2\ km/s, \quad V_0 = 0, \quad V(t) = 11.2\ km/s, \quad g = 9.8\ m/s^2, \quad t - t0 = 100s.$$

Solving for M_0/M shows that

$$\frac{M_0}{M} = 441$$

when escape velocity occurs. So, the rocket needs to carry over 400 times more fuel than payload!