Physics 3550

Newton's Second Law.

Relevant Sections in Text: §1.4 – 1.7, 2.5

Newton's Second Law.

Needless to say, Newton's second law holds only in inertial reference frames. We have:

Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

This is of course, the famous "$F = ma$", once we interpret "alteration of motion" as "acceleration" and combine "motive force" and "direction of the right line" into the force vector. Apart from examples and applications, which we will get to shortly, there is a lot going on "under the hood" in Newton's second law which we need to discuss.

First of all, let us deal with a common mental mistake made by beginners. It is perhaps tempting to read Newton's second law as a literal identification of force with mass times acceleration. This is not the way to think of it. Think of it as follows. You are in the lab. You have set up an experiment, say, where you have connected a spring to a body (modeled as a "particle", if you like). You displace the mass from equilibrium, let it go, and measure the position as a function of time, so that you now have access to velocity as a function of time and acceleration as a function of time, too. You can measure the acceleration quite directly; that's the point. The relation of acceleration to force is an idea, a proposal of Newton's which, *a priori*, may or may not be true. Anyway, you now measure $\vec{r}(t)$ — and hence $\vec{a}(t)$ — for a variety of bodies with a variety of initial conditions. What you find is that the accelerations (as a function of time) for different bodies are, in general, different. Fine. The observation of Newton is, though, that associated to each body there is a single constant – the mass – which when multiplied by the acceleration vector stems from a single object: a vector *field* – the force – which is the same no matter which body is used. Because this quantity is independent of the body (in the sense just described) we ascribe to it a reality of its own and postulate that this fact holds universally – Newton's second law. This force, so defined, will in general vary in position, velocity, and in time (as does the acceleration) – this has to be made considerably more precise, which we shall do momentarily. The idea here is that we can take the point of view that all the bodies are being influenced by a single entity – the force – and this gives rise to a wide variety of accelerations, depending upon the mass of the body, its position, and so forth. The acceleration is a property of a given body's state (of motion) and is only defined on a given curve. The mass is an intrinsic property of the body. The force is an external *field* quantity characterizing the environment and the body's interaction with it. Don't think of "$F = ma$" as an identification of two quantities. Think of it as expressing the response (acceleration) of a body to an external influence (force). The right hand side of $F = ma$ refers to the body; the left hand side refers to its environment.

With all this conceptual stuff in mind, we can give a much more precise statement of the second law using the language of physics: mathematics. Given a curve $\vec{r}(t)$ depicting the motion of a body obeying Newton's second law, we have

$$\vec{F}(\vec{r}(t), \vec{v}(t), t) = m\vec{a}(t).$$

I have been very pedantic with the notation here. Much more pedantic than you have probably seen before. The usual, much cleaner looking version of the formula: $\vec{F} = m\vec{a}$, is perfectly okay if you already know what you are doing. It can be disastrous if you don't. The idea of this formula is that you have a curve, $\vec{r}(t)$, representing the motion of a particle. We then have, of course, three functions of time representing the velocity and three more functions of time representing the acceleration of the body at time t:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}, \quad \vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2},$$

as we have discussed. Now for the force. The force is, *a priori*, viewed as a *three-dimensional vector field in a seven-dimensional space!* That is, it is a vector in Euclidean space which depends upon (in general) position, velocity, and time. This is best explained via examples.

First, let us consider the mass on a spring alluded to above. Ignore all forces except the force due to the spring for simplicity. Orient the x-axis along the spring. The force is

$$\vec{F} = -k\,x\,\hat{i}.$$

In this case the force depends only upon position: $\vec{F} = \vec{F}(\vec{r}) = \vec{F}(x, y, z)$ – in fact, it only depends upon x in our chosen coordinates, $\vec{F} = \vec{F}(x)$. In any case, you can see that the force is certainly a vector *field* (a family of vectors varying with position) instead of just a single vector.

Let's look at another example. Let the body be a particle of charge q moving in a uniform magnetic field $\vec{B}$. Pick the z axis to be along $\vec{B}$ so that $\vec{B} = B\hat{k}$. The force on the particle is (from the Lorentz force law):*

* Make sure you can derive this last equality!

$$\vec{F} = q\vec{v} \times \vec{B} = qB\left(v^y\,\hat{\imath} - v^x\,\hat{\jmath}\right).$$

Here $\vec{v}$ is, of course, the velocity of the particle. You see in this case that $\vec{F} = \vec{F}(\vec{v}) = \vec{F}(v^x, v^y)$. As you know, the magnetic field usually depends upon position – it's a vector *field* after all. In this case we have, of course, $\vec{F} = \vec{F}(\vec{r}, \vec{v})$. For example, if the uniform field used above grows linearly with z, say,

$$\vec{B} = \beta z\hat{k},$$

we have

$$\vec{F}(\vec{r}, \vec{v}) = q\beta z\left(v^y\,\hat{\imath} - v^x\,\hat{\jmath}\right).$$

Here β is a constant.

As a third example, lets consider time-dependent forces – we are focusing on the third argument of $\vec{F}(\vec{r}, \vec{v}, t)$. Consider an electron in a sinusoidally oscillating uniform electric field,

$$\vec{E} = E\cos(\omega t)\,\hat{\imath}.$$

The force on the electron (viewed as a particle, of course) is

$$\vec{F} = q\vec{E} = qE\cos(\omega t)\,\hat{\imath}.$$

You see that t sitting there inside the cosine? That's the t in the general notation $\vec{F} = \vec{F}(t)$.

Finally, I think you can see how all 7 variables can be used to determine $\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$. Just consider an electron moving in the combined electromagnetic field described above!

To write down Newton's second law we take $\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$ and evaluate it on the curve $\vec{r} = \vec{r}(t)$ of interest. This means that everywhere we make the substitution

$$\vec{r} \longrightarrow \vec{r}(t), \quad \vec{v} \longrightarrow \frac{d\vec{r}(t)}{dt}.$$

In particular this is how we get the notation I used originally $\vec{F} = \vec{F}(\vec{r}(t), \vec{v}(t), t)$.

Let me introduce one more idea and some terminology. You can see that the left hand side of "$F = ma$" can, in general, change in time. Time varying forces are all around us, so that's no surprise. But, mathematically, it's important to notice that the time dependence arises in two distinct ways. We have $\vec{F} = \vec{F}(\vec{r}(t), \vec{v}(t), t)$ where the t appears via the curve (position and velocity) and as third argument all by itself. The former kind of time dependence is called *implicit* while the latter is called *explicit*. Physically it is not too hard to see what is going on. Explicit time dependence happens when the force is itself changing in time, *e.g.,* some mad scientist is turning the knobs and throwing switches. Our oscillating electric field example above is a good illustration of explicit time dependence in the force. The force vector changes in time because there is a "t" in the formula for it. Explicit time dependence is determined by the outside world, independently of the particle. Implicit time dependence occurs because of two facts: (1) the force depends upon the particle's position and/or velocity, (2) the particle position and/or velocity is changing in time. For example, in the harmonic oscillator example, the force vector field we use is the same for all time, there is no explicit time dependence in

$$\vec{F} = -k\,x\,\hat{\imath}.$$

But, the force experienced by the body *does* depend upon time because as the body moves we have $x = x(t)$ and the force changes with the particles displacement from equilibrium. For example, a possible motion of the system is

$$x = \cos(\omega t), \quad y = 0, \quad z = 0, \quad \omega = \sqrt{\frac{k}{m}},$$

in which case the force experienced by the particle as it moves is

$$\vec{F}(\vec{r}(t)) = -k\cos(\omega t)\hat{\imath}.$$

The equations of motion.

The relationship between the particle's motion, $\vec{r} = \vec{r}(t)$, and the applied force can be viewed as a differential equation. Mathematically we view the force as some given function of position, velocity and time, $\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$. Given a curve $\vec{r}(t)$ we evaluate the force on the curve to get, $\vec{F} = \vec{F}(\vec{r}(t), \vec{v}(t), t)$. We equate this to mass times acceleration and we get the differential equation:

$$m\frac{d^2\vec{r}(t)}{dt^2} - \vec{F}(\vec{r}(t), \frac{d\vec{r}(t)}{dt}, t) = 0.$$

More precisely, we get (in general) a coupled system of 3 second-order, quasi-linear ordinary differential equations. These are called, naturally enough, the *equations of motion.*

Usually, the job of the physicist is to (1) figure out the equations of motion for the system of interest, (2) try to figure out as much as possible about the curve $\vec{r} = \vec{r}(t)$ given the equations of motion. But another very important task would be to run this scenario backwards: characterize the force given (experimentally determined) curves. I should point out that only for the simplest models of physical systems can you actually find a closed form expression for the solution $\vec{r}(t)$ of the equations of motion. It's not that there isn't a solution (see below), it's that the motion is too complicated for any simple elementary functions to describe it. Often times one resorts to numerical work to extract information about $\vec{r}(t)$.

The mathematicians have given us some very important *a priori* information about solutions to the equations of motion for any choice of the force. First of all, solutions always exist (at least for some time interval). Good. Secondly, the solutions are not unique in general. There are many solutions. This is also good because, given a force field, we know that infinitely many motions are possible. Finally, the solutions are uniquely determined by their initial position and initial velocity. This is a strong physical prediction of Newtonian mechanics, one which is readily born out by experience and experiment.

As an example of equations of motion, let us return to the charged particle moving in the uniform magnetic field, as described earlier. We get

$$m\frac{d^2x(t)}{dt^2} - qB\frac{dy(t)}{dt} = 0$$

$$m\frac{d^2y(t)}{dt^2} + qB\frac{dx(t)}{dt} = 0$$

$$m\frac{d^2z(t)}{dt^2} = 0.$$

In this case, because the magnetic field does not depend upon position the equations are all linear. Moreover, because the magnetic field does not depend upon time the equations have constant coefficients. These simplifying features mean we can expect to write down the curve $\vec{r} = \vec{r}(t)$ which satisfies these equations. Here's how...

Motion in a uniform magnetic field

As an example of finding the solution to equations of motion, we attack the fundamental and important case of a charged particle in a uniform magnetic field. We refer to the equations of motion provided above. Firstly, the last equation for $z(t)$ is *decoupled* and is easily solved:

$$z(t) = z_0 + v_0^z t.$$

Here z_0 and v_0^z are integration constants which are fixed by initial conditions. The other two equations, which only involve x and y, are solved as follows. First note that neither of (x, y) appear undifferentiated. So, we make a simple change of variables,

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}.$$

The equations of motion for x and y are now

$$m\frac{du}{dt} - qBv = 0, \quad m\frac{dv}{dt} + qBu = 0.$$

Differentiate the first equation and substitute in for $\frac{dv}{dt}$ from the second equation to get:

$$\frac{d^2u}{dt^2} + \left(\frac{qB}{m}\right)^2 u = 0.$$

Mathematically, this is just the equation for a harmonic oscillator where u is the displacement from equilibrium, and the angular frequency is given by

$$\omega = \frac{qB}{m}.$$

Therefore, the solution is easily seen to be

$$u(t) = A\cos(\omega t) + B\sin(\omega t),$$

where A and B are constants. With $u(t)$ determined, we can determine $v(t)$ from the equations of motion to be

$$v(t) = \frac{m}{qB}\frac{du}{dt} = -A\sin(\omega t) + B\cos(\omega t).$$

After all this maneuvering it is a good idea to confirm that u and v do solve their differential equations. I'll leave that to you. We can now go back to the x and y variables. This just means integrating u and v with respect to time, respectively, and adding in an integration constant each time. All we have are cosines and sines, so this is easily done and I will leave the details to you. The result is readily seen to be of the form:

$$x(t) = \alpha\cos(\omega t) + \beta\sin(\omega t) + \gamma$$

$$y(t) = \beta\cos(\omega t) - \alpha\sin(\omega t) + \delta.$$

Here $\alpha, \beta, \gamma, \delta$ are all constants. Recalling the solution for $z(t)$, I think you can see from the form of the solutions for $x(t)$ and $y(t)$ that the motion is helical around an axis parallel to the z axis. Indeed, we have already seen that motion in z is at constant speed, and from the above solution for x and y we have

$$(x(t) - \gamma)^2 + (y(t) - \delta)^2 = \alpha^2 + \beta^2.$$

Thus the x–y motion is circular with radius $\alpha^2 + \beta^2$, centered about (γ, δ).

Notice that the solutions we have found are parametrized by 6 constants, which I called $(z_0, v_0^z, \alpha, \beta, \gamma, \delta)$. Since this number (6) is the same as the number of parameters needed to specify the initial position and velocity, it won't surprise you to find that these 6 constants can be determined by the initial conditions. This is, of course, in accord with general results from the theory of differential equations. The 6 constants parametrize all possible motions of a charged particle in a uniform magnetic field which are in accord with Newton's second law.

Exercise: Express $(z_0, v_0^z, \alpha, \beta, \gamma, \delta)$ *in terms of the initial conditions. What values of these constants correspond to uniform motion parallel to the* z *axis? What values of these constants correspond to circular motion in the* x*-*y *plane?*

The superposition principle.

There is a tacit assumption we make when using Newton's laws that really should get a law all its own. I call it the *superposition principle.* This is just the postulate, already well-known by you, that when two or more forces act on a particle, the resulting net force is just the vector sum of the individual force vectors:

$$\vec{F}_{total} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i.$$

I tell you this, first of all, to remind you of this rule. Second of all, I tell you this to emphasize that it, too, is a part of the laws of Newton and is subject to experimental verification. It need not be true that forces combine as vectors, or even that forces combine in a linear fashion. In fact, there are situations where the superposition principle is not valid. For example, strong gravitational fields don't behave this way. As usual, for a wide range of physical situations the superposition principle *does* hold with extremely good accuracy, so we take it as given.