The Dirac Delta Function

Overview and Motivation: The Dirac delta function is a concept that is useful throughout physics. For example, the charge density associated with a point charge can be represented using the delta function. As we will see when we discuss Fourier transforms (next lecture), the delta function naturally arises in that setting.

Key Mathematics: The Dirac delta function!

I. Introduction

The basic equation associated with the **Dirac delta function** $\delta(x)$ is

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0), \tag{1}$$

where $f(x)$ is any function that is continuous at $x = 0$. Equation (1) should seem strange: we have an integral that only depends upon the value of the function $f(x)$ at $x = 0$. Because an integral is "the area under the curve," we expect its value to not depend only upon one particular value of x. Indeed, there is no function $\delta(x)$ that satisfies Eq. (1). However, there is another kind of mathematical object, known as a *generalized function* (or *distribution*), that can be defined that satisfies Eq. (1).

A **generalized function** can be defined as the limit of a sequence of functions. Let's see how this works in the case of $\delta(x)$. Let's start with the normalized Gaussian functions

$$g_n(x) = \sqrt{\frac{n}{\pi}} e^{-nx^2} . \tag{2}$$

Here $n = 1/\sigma^2$, where σ is the standard Gaussian width parameter. These functions are normalized in the sense that their integrals equal 1,

$$\int_{-\infty}^{\infty} g_n(x) dx = 1 \tag{3}$$

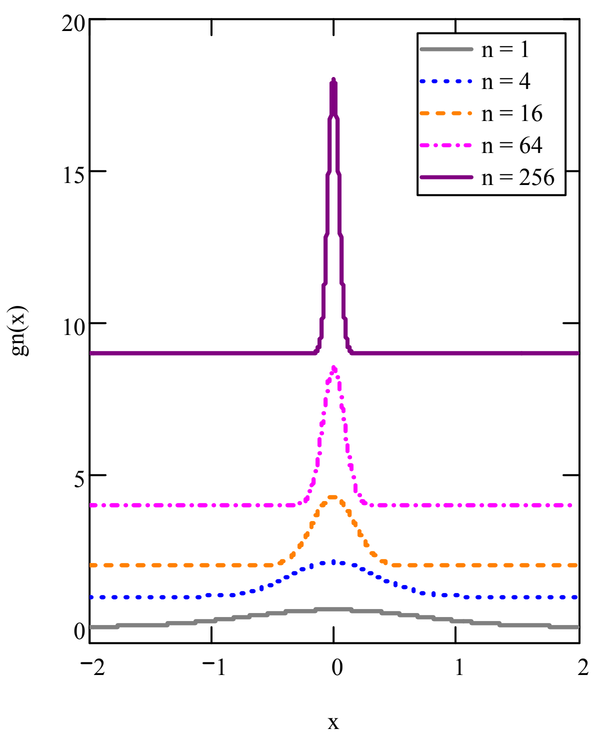
for any value of n (>0). Let's now consider the sequence of functions for $n = 1, 2, 3, \ldots$,

$$g_1(x)=\sqrt{\frac{1}{\pi}}e^{-x^2},\ g_2(x)=\sqrt{\frac{2}{\pi}}e^{-2x^2},\ \ldots,\ g_{100}(x)=\sqrt{\frac{100}{\pi}}e^{-100x^2},\ \ldots \tag{4}$$

What does this sequence of functions look like? We can summarize this sequence as follows. As n increases

(a) $g_n(0)$ becomes larger;
(b) $g_n(x\neq 0)$ eventually becomes smaller;
(c) the width of the center peak becomes smaller;
(d) but $\int_{-\infty}^{\infty} g_n(x)dx = 1$ remains constant.

The following figure plots some of the functions in this sequence.



Let's now ask ourselves, what does the $n=\infty$ limit of this sequence look like? Based on (a) through (d) above we would (perhaps simplistically) say

(a) $g_\infty(0)=\infty$;
(b) $g_n(x\neq 0)=0$;
(c) the width of the center peak equals zero;

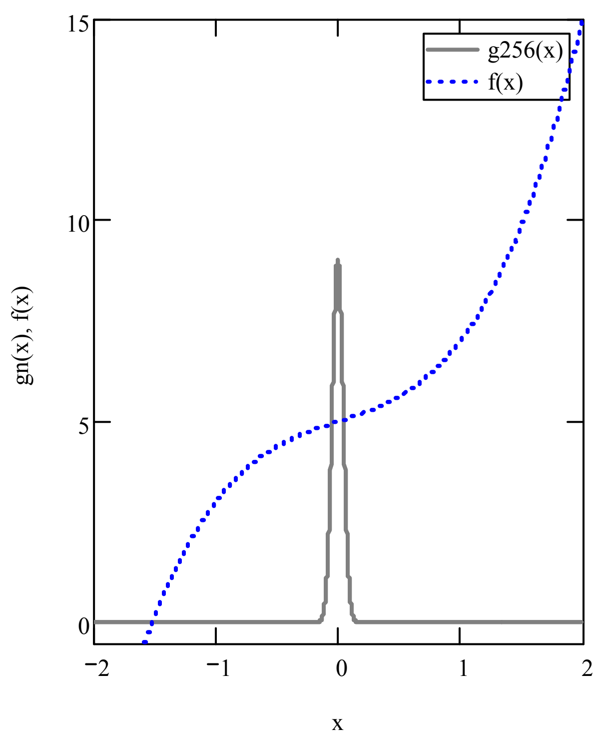
(d) but $\int_{-\infty}^{\infty} g_\infty(x)dx = 1$.

Note that (a) and (b) are not compatible with (d) if $g_\infty(x)$ is a function in the standard sense, because for a function (a) and (b) would imply that the integral of $g_\infty(x)$ is zero.

So how should we think of this sequence of functions, then? Well, the sequence is only really useful if it appears as part of an integral, as in, for example,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} g_n(x) f(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2} f(x)dx . \tag{5}$$

Let's calculate the integral, and then the limit in Eq. (5). The following figure should help with the calculation.



As n get large, $g_n(x) = \sqrt{\tfrac{n}{\pi}}e^{-nx^2}$ becomes narrower such that it only has weight very close to $x = 0$. Thus, as far as the integral is concerned, for large enough n only $f(x)$ at $x = 0$ is important. We can thus replace $f(x)$ by $f(0)$ in the integral, which gives us

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\ f(x)dx = f(0)\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\ dx = f(0)\lim_{n\to\infty}1 = f(0) \tag{6}$$

Thus, the integral on the lhs of Eq. (1) is really shorthand for the integral on the lhs of Eq. (6), That is, the Dirac delta function is defined via the equation

$$\int_{-\infty}^{\infty}\delta(x)f(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}\ f(x)dx \tag{7}$$

Now often (as physicists) we often get lazy and write

$$\delta(x) = \lim_{n\to\infty}\sqrt{\tfrac{n}{\pi}}e^{-nx^2}, \tag{8}$$

but this is simply shorthand for Eq. (7). Eq. (8) really has no meaning unless the function $\sqrt{\tfrac{n}{\pi}}e^{-nx^2}$ appears *inside* an integral and the limit $\lim_{n\to\infty}$ appears *outside* the same integral. However, after you get used to working with the delta function, you will rarely need to even think about the limit that is used to define it.

One other thing to note. This particular sequence of functions $g_n(x) = \sqrt{\tfrac{n}{\pi}}e^{-nx^2}$ that we have used here is not unique. There are infinitely many sequences that can be used to define the delta function. For example, we could also have defined $\delta(x)$ via

$$\delta(x) = \lim_{n\to\infty}\frac{1}{\pi}\frac{\sin(nx)}{x}. \tag{9}$$

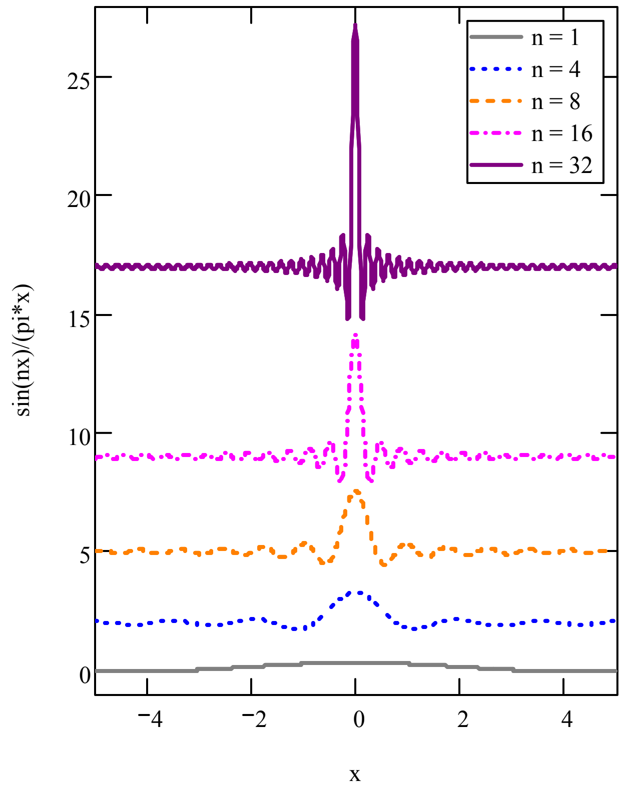
The sequence of functions $\sin(nx)/(\pi x)$ is illustrated in the figure at the top of the next page.

Notice that the key features of both of these two difference sequences are expressed by (a) – (d) at the top of page 5.

II. Delta Function Properties

There are a number of properties of the delta function that are worth committing to memory. They include the following,

$$\int_{-\infty}^{\infty}\delta(x-x')f(x)dx = f(x'), \tag{10}$$



$$\int_{-\infty}^{\infty} \delta'(x) f(x) dx = -f'(0) \tag{11}$$

$$\delta(x/a) = |a|\,\delta(x) \tag{12}$$

The proof of Eq. (10) is relatively straightforward. Let's change the integration variable to $y = x - x'$, $dy = dx$, which gives

$$\int_{-\infty}^{\infty} \delta(x - x') f(x) dx = \int_{-\infty}^{\infty} \delta(y) f(y + x') dy \,. \tag{13}$$

Then using Eq. (1), we see that Eq. (10) is simply equal to $f(x')$. *QED*.

Let's also prove Eq. (12). We do this in two steps, for $a > 0$ and then for $a < 0$.
(*i*) First, we assume that $a > 0$. Then

$$\int_{-\infty}^{\infty} \delta(x/a)\,dx = \int_{-\infty}^{\infty} \delta(x/|a|)\,dx \tag{14}$$

Changing integration variable $y = x/|a|$, $dy = dx/|a|$, this last equation becomes

$$\int_{-\infty}^{\infty} \delta(x/a)\,dx = |a|\int_{-\infty}^{\infty} \delta(y)\,dy \tag{15}$$

and changing variables back to x via $x = y$, $dx = dy$ gives

$$\int_{-\infty}^{\infty} \delta(x/a)\,dx = |a|\int_{-\infty}^{\infty} \delta(x)\,dx \tag{16}$$

and so for $a > 0$ we have $\delta(x/a) = |a|\delta(x)$.

(*ii*) We now assume $a < 0$. Then we have

$$\int_{-\infty}^{\infty} \delta(x/a)\,dx = \int_{-\infty}^{\infty} \delta(-x/|a|)\,dx \tag{17}$$

Changing integration variable $y = -x/|a|$, $dy = -dx/|a|$, this last equation becomes

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x/a)\,dx &= -|a|\int_{\infty}^{-\infty} \delta(y)\,dy \\ &= |a|\int_{-\infty}^{\infty} \delta(y)\,dy \\ &= |a|\int_{-\infty}^{\infty} \delta(x)\,dx\end{aligned} \tag{18}$$

and so for $a < 0$ we also have $\delta(x/a) = |a|\delta(x)$. *QED*.

We leave the proof of Eq. (11) as an exercise.

III. Fourier Series and the Delta Function

Recall the complex Fourier series representation of a function $f(x)$ defined on $-L \le x \le L$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L} \, , \tag{19a}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \, e^{-in\pi x/L} \, dx \, . \tag{19b}$$

Let's now substitute c_n from Eq. (19b) into Eq. (19a). Before we do this we must change the variable x in either Eq. (19a) or (19b) to something else because the variable x in Eq. (19b) is just a (dummy) integration variable. Changing x to x' in Eq. (19a) and doing the substitution we end up with

$$f(x') = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(x) \, e^{-in\pi x/L} \, dx \right) e^{in\pi x'/L} \tag{20}$$

Let's now switch the integration and summation (assuming that this is OK to do). This produces

$$f(x') = \int_{-L}^{L} \left[\frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{in\pi(x'-x)/L} \right] f(x) \, dx \tag{21}$$

If we now compare Eq. (21) to Eq. (10), we see that we can identify another representation of the delta function

$$\delta(x - x') = \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{in\pi(x'-x)/L} \tag{22}$$

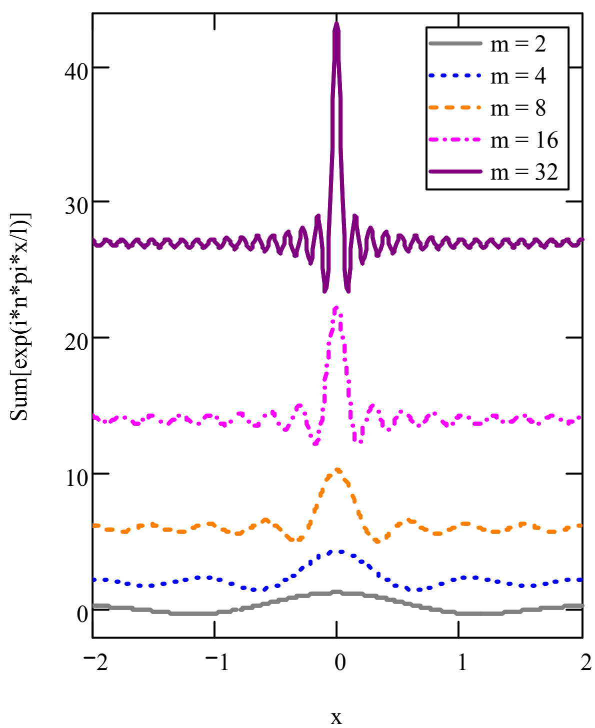
or setting $x' = 0$ we have

$$\delta(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{-in\pi x/L} \tag{23}$$

So how is this equation related to the delta function being defined as the limit of a sequence of functions? Well, we can re-express Eq. (23) as a sequence of functions via

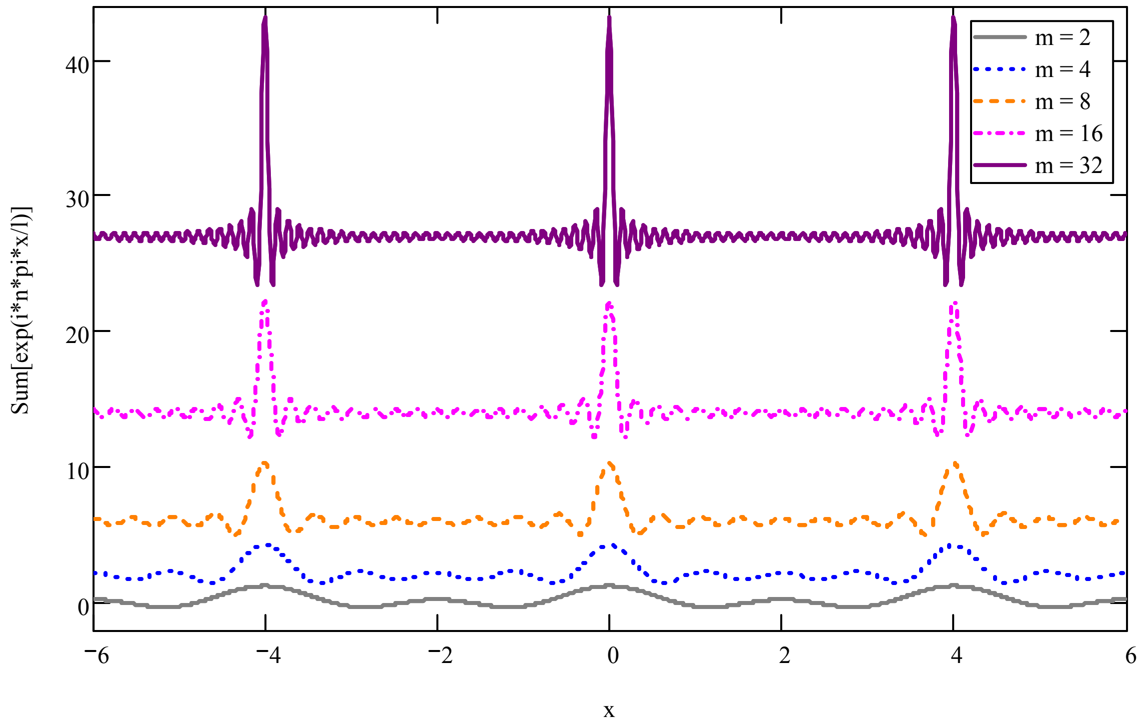
$$\delta(x) = \lim_{m\to\infty} \frac{1}{2L} \sum_{n=-m}^{m} e^{-in\pi x/L} \ . \tag{24}$$

The following figure plots $\frac{1}{2L}\sum_{n=-m}^{m} e^{-in\pi x/L}$ for several values of m (for $L = 2$). Notice that these functions are quite similar to the function $\sin(mx)/(\pi x)$ plotted above.[1]



However, there is one important difference between these two sequences of functions. Because the functions $e^{-in\pi x/L}$ that appear in the sum in Eqs. (23) and (24) all repeat with on an interval of length $2L$, Eq. (24) is actually a series of delta functions, centered at $x = 0, \pm 2L, \pm 4L, \pm 6L, \ldots$. This is illustrated in the next figure, where we have expanded the x axis beyond the limits of $-L$ to L. Thus, the equality expressed by Eq. (23) or (24) is only valid on the interval $-L \le x \le L$.

[1] We have changed the n to m in the $\sin(nx)/(\pi x)$ functions because we are now using m to label the functions in the sequence.



Exercises

*$\mathbf{15.1}$ Equation (11), $\int_{-\infty}^{\infty}\delta'(x)f(x)dx = -f'(0)$, can be taken as the definition of the derivative of the delta function. Treating the delta function as a normal function, show that Eq. (11) is true. (Hint: use integration by parts.)

*$\mathbf{15.2}$ Show that the equation $\delta(x) = \lim_{m\to\infty}\frac{1}{2L}\sum_{n=-m}^{m} e^{-in\pi x/L}$ can be re-expressed as $\delta(x) = \lim_{m\to\infty}\frac{1}{2L}\sum_{n=-m}^{m}\cos\left(\frac{n\pi x}{L}\right)$. This is perhaps more appealing because the delta function is a real function and the rhs is now explicitly real.

*$\mathbf{15.3}$ Find another sequence of functions, not based on either the Gaussian or $\sin(x)/x$ functions, that has as its limit the delta function.