# I. THE IMPORTANCE OF DRAG FORCES

In introductory discussions of an object moving through the Earth's atmosphere, the resistance of the air on the motion of the object is usually neglected. The main reason for this neglect is that the **equation of motion** for the object (Newton's second law) is then fairly simple,

$$m\vec{a} = -mg\,\hat{y}.\tag{1}$$

This equation of motion includes only the gravitational force (magnitude = mg). Because of the simplicity of Eq. (1), one can exactly solve for the position  $\vec{r}(t)$  and velocity  $\vec{v}(t)$  of the object. As it turns out, the path of an object subject to this equation of motion is a parabola, as illustrated in Fig. 1. The solid curve in Fig. 1 is the path that a baseball would take if it left Sammy Sosa's bat with an initial speed of 150 mi/hr at an initial angle of 45 degrees with respect to the horizontal. Notice that the ball is calculated to travel a distance of 1600 feet! Even Sammy would have a difficult time actually hitting a baseball such a distance!

What's wrong with the calculation? As you have probably guessed, the neglect of air resistance results in a significant overestimation of the distance the baseball travels. If the appropriate force due to air resistance is included in the calculation of the flight of the baseball, then the dashed curve in Fig. 1 results. The result is still quite a long home run (or long foul ball), but we now have a realistic distance (500 ft) for one of Sammy's hits. Drag forces, of which air resistance is one example, are generally important when a solid object moves through a fluid, either a liquid or gas. Other examples include a bicyclist riding a bike, a boat moving through water, or the analytical technique of electrophoresis.

Here we discuss two equations that are often used to describe drag forces. Keep in mind that the equations for these forces are not exact in the sense that, for example, Newton's law of gravitational force is. This is because they describe, with only a few parameters, the interaction of atoms at the rather complicated interface between the solid and the fluid. Still, the equations are quite useful in a variety of circumstances.

## **II. LOW VELOCITY, HIGH VISCOSITY**

Let's now consider an object moving at relatively slow velocity (slow will be defined later) through a fairly **viscous** (think sticky!) medium. A good example might be a small metallic sphere, such as a BB, moving through molasses. For the moment we will neglect gravity and assume that the object is subject only to the drag force. A good approximation to the drag force (of the viscous medium) on the solid object is

$$\vec{F}_1 = -K_1 \vec{v} = -K_1 v \, \hat{v}. \tag{2}$$

Here  $K_1$  is a *positive* constant that depends upon both the object and the fluid through which it moves. What does this equation tell us? It simply says that the drag force  $F_1$  is proportional to the velocity and that the force points in the direction opposite to the velocity, so that the force always opposes the motion of the object. (Recall that  $\hat{v}$  is a unit vector in the direction of the velocity.)

This equation for the force is most ideally suited to describing an object with circular cross section moving through the medium. For such an object the constant  $K_1$  is equal to

$$K_1 = C_1 \pi a \eta, \tag{3}$$

where  $C_1$  is another constant, a is the radius of the object (perpendicular to the direction of motion) and  $\eta$  is the viscosity of the medium. (The units of viscosity are kg m<sup>-1</sup> s<sup>-1</sup> or equivalently N s m<sup>-2</sup>.) For a sphere  $C_1 = 6$ . Equation (3) explicitly shows how the drag force depends on the size of the object and the medium through which it moves. For objects that are not circular in cross section Eq. (3) can still be used to good approximation. However, the parameter a will instead be some distance roughly half the size of the object.

#### III. HIGH VELOCITY, LOW VISCOSITY

Another situation commonly encountered is an object moving fairly fast through a low-viscosity medium. The drag force then takes a form different than Eq. (2). An example of such motion is a baseball moving through the air. In this case the drag force is better represented as

$$\vec{F}_2 = -\frac{1}{2}K_2 v^2 \,\hat{v},\tag{4}$$

where  $K_2$  is a positive constant that again depends upon properties of the solid object and the fluid. This equation says that the drag force is again opposite to the direction of the velocity, but this time the force is proportional to the *square* of the velocity, as opposed to being linear in the velocity. In this case the constant  $K_2$  is usually expressed as

$$K_2 = C_D S \rho. \tag{5}$$

The parameter  $C_D$  is known as the **drag coefficient** and is typically close to 0.5, but it does depend upon the

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Figure 1. Flights of baseballs hit by Sammy Sosa, first without air resistance (solid line) and with air resistance (dotted line).

exact shape of the object. The parameter S is the crosssectional area (perpendicular to the direction of travel) of the object (units = m<sup>2</sup>), and  $\rho$  is the density (kg/m<sup>3</sup>) of the medium. For an object with circular cross section (such as a sphere) Eq. (5) can be written as

$$K_2 = C_D \pi a^2 \rho, \tag{6}$$

where a is the radius of the cross section.

Equation (4) with  $K_2$  given by Eq. (6) was used to describe the drag force in the more realistic calculation of the baseball flight shown in Fig. 1.

# IV. SO WHICH DRAG FORCE SHOULD I USE?

Because all normal fluids have both density and viscosity, both types of drag force act on any object moving through a fluid. However, in many cases we only have to use one of the forces to describe the drag. To see how this works we have made several graphs of the magnitude of each force vs the speed v of an object, shown in Fig. 2. Notice that for very low velocities (part (a) of the figure – check the x-axis scale) the linear v drag force (solid line) is much larger than the quadratic v force (dotted line), but at very large velocities the  $v^2$  force becomes significantly larger than linear v force, as shown in part (c). However, there is always one speed where the two forces are equal. From Eqs. (2) and (4) it can be shown the two forces are equal at a speed of

$$v_{cross} = \frac{2K_1}{K_2},\tag{7}$$

which is known as the **crossover speed**.

If we use Eqs. (3) and (6) for  $K_1$  and  $K_2$  in (7) then for a sphere we can express the crossover speed as

$$v_{cross} = \frac{12\eta}{C_D a \rho}.$$
(8)

So, if the speed of the object is much greater than  $v_{cross}$ , then  $\vec{F_2}$  can be used to describe the drag; if the speed is much less than  $v_{cross}$ , then  $\vec{F_1}$  can be used. However, if the velocity v is close to  $v_{cross}$ , then the sum  $\vec{F_1} + \vec{F_2}$  must be used to accurately describe the drag on the object.

For one of your homework problems you will show for a baseball moving through air that the crossover speed is much less than any typical speed important to the game of baseball. Thus the  $v^2$  force can be used to describe the drag on the home-run balls that come off of Sammy's bat.

## V. GRAVITY AND DRAG FORCES

Let's now consider the case (as in the lab with the falling coffee filter) where an object is moving only in the vertical direction, subject to both the force of gravity and an appropriate drag force. If we release the object from rest then initially the only force on it is that due to gravity. However, as the object speeds up the drag force increases (from zero) until its magnitude equals the force due to gravity. Because the gravitational and drag forces point in opposite directions, the net force on the object is zero. Thus the object no longer accelerates (its velocity remains constant), and it falls at a constant rate. This is the condition of **terminal velocity** 

Setting the force of gravity equal to the drag force (in the general case where we must include both kinds of drag forces) gives us an equation that determines the terminal velocity  $v_{term}$  of a freely falling object,

$$mg = K_1 v_{term} + \frac{1}{2} K_2 v_{term}^2.$$
 (9)



Figure 2.  $F_1(v)$  (solid line) and  $F_2(v)$  (dotted line) vs speed v. Here  $K_1 = 1$  N s/m and  $K_2 = 1$  N (s/m)<sup>2</sup>. The two forces are equal at the crossover velocity  $v_{cross} = 2$  m/s.

If the object is a sphere, then we can use Eqs. (3) and (6) and substitute for  $K_1$  and  $K_2$  in Eq. (9), which yields

$$mg = 6\pi\eta(av_{term}) + \frac{\pi}{2}C_D\rho(av_{term})^2.$$
 (10)

As discussed above, however, we often only need to include one of the drag force terms. For example, if the linear v drag force is the important drag force, then the second term on the right-hand-side of Eq. (9) can be neglected. Solving Eq. (9) for the terminal velocity then results in

$$v_{term} = \frac{mg}{K_1}.$$
 (11)

If the  $v^2$  drag force is the dominant drag force, then the first term on the right-hand-side of Eq. (9) can be neglected, resulting in

$$v_{term} = \sqrt{\frac{2mg}{K_2}}.$$
 (12)

# VI. ELECTROPHORESIS

The technique of electrophoresis is often used to identify proteins or DNA sequences. The basis for this technique lies in the difference in drag force that different molecules undergo in a fluid. In the technique the sample under investigation (a particular protein, say) is placed at one end of the electrophoresis cell, which contains the medium that provides the drag force. An electric field is applied that quickly accelerates the (charged) molecules to terminal velocity. For a molecule with charge q (unit = Coulomb = C) the electric force on the molecule is given by

$$\vec{F}_{elec} = q\vec{E},\tag{13}$$

where  $\vec{E}$  is the electric field (units = N C<sup>-1</sup>). So instead of gravity and the drag force balancing it is now the electric force and the drag force that balance each other. The terminal velocity depends on the size (and thus the molecular weight) of the molecule. The molecules are allowed to drift for a certain amount of time. From the distance that they travel during that time their molecular weight can then be identified.