Neutron Interference

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1 Magnetic moment

Consider the interaction of a spin-$\frac{1}{2}$ particle with a magnetic field. The Hamiltonian is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$

where we take the magnetic field $\mathbf{B} = B\mathbf{k}$ to be uniform, constant, and in the z-direction. The magnetic moment of a particle is proportional to its angular momentum, so as an operator it becomes

$$\begin{aligned} \hat{\boldsymbol{\mu}} &= \frac{ge}{mc}\hat{\mathbf{S}} \\ &= \frac{ge\hbar}{2mc}\hat{\boldsymbol{\sigma}} \end{aligned}$$

where the "g factor" is very close to 2, with $m = m_e$ for the electron; for the neutron we have $g_n \approx -1.91$ and $m = m_p$ (yes, proton mass) where $\mu_N = \frac{e\hbar}{2m_p c}$ is called the nuclear magneton. Therefore, for a neutron in the magnetic field,

$$\begin{aligned} \hat{H} &= -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} \\ &= -\frac{g_n e}{m_p c}\hat{\mathbf{S}} \cdot \mathbf{B} \\ &= -\frac{g_n e B}{m_p c}\hat{S}_z \end{aligned}$$

We define a frequency,

$$\omega \equiv \frac{g_n e B}{m_p c} > 0$$

By studying this system we can check experimentally that spinors rotate at half the rate of vectors.

2 Rotations

Consider a rotation by an angle φ about an axis along $\mathbf{n}$. The rotation operator that accomplishes this is

$$\mathcal{U} = e^{\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}$$

If this is used to rotate a spinor,

$$|\chi\rangle = a\,|+\rangle + b\,|-\rangle$$

we have

$$\begin{aligned} |\chi'\rangle &= \mathcal{U}\,|\chi'\rangle \\ &= e^{\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}\left(a\,|+\rangle + b\,|-\rangle\right) \end{aligned}$$

For a rotation around the z-axis, $e^{\frac{i\varphi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}} = e^{\frac{i\varphi}{2}\sigma_z} = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix}$ and this becomes

$$|\chi'\rangle = ae^{\frac{i\varphi}{2}}|+\rangle + be^{-\frac{i\varphi}{2}}|-\rangle$$

After a 2π rotation,

$$\begin{aligned} |\chi'\rangle &= ae^{\frac{2\pi i}{2}}|+\rangle + be^{-\frac{2\pi i}{2}}|-\rangle \\ &= -\left(a|+\rangle + b|-\rangle\right) \\ &= -|\chi\rangle \end{aligned}$$

and the spinor has rotated only halfway around. It returns to itself only after 4π. By contrast, if we rotate a 3-vector, only a 2π rotation is required. For example, The spin vector, $\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}$, rotates under the same rotation according to

$$\begin{aligned} \hat{\mathbf{S}}' &= \mathcal{U}\hat{\mathbf{S}}\mathcal{U}^\dagger \\ &= \frac{\hbar}{2}e^{\frac{i\varphi}{2}\sigma_z}\hat{\boldsymbol{\sigma}}e^{-\frac{i\varphi}{2}\sigma_z} \\ &= \frac{\hbar}{2}\left(1\cos\frac{\varphi}{2} + i\sigma_z\sin\frac{\varphi}{2}\right)\hat{\boldsymbol{\sigma}}\left(1\cos\frac{\varphi}{2} - i\sigma_z\sin\frac{\varphi}{2}\right) \end{aligned}$$

so that the components are given by

$$\begin{aligned} \hat{S}'_x &= \frac{\hbar}{2}\left(1\cos\frac{\varphi}{2} + i\sigma_z\sin\frac{\varphi}{2}\right)\sigma_x\left(1\cos\frac{\varphi}{2} - i\sigma_z\sin\frac{\varphi}{2}\right) \\ &= \frac{\hbar}{2}\left(\sigma_x\cos^2\frac{\varphi}{2} + i\left[\sigma_z,\sigma_x\right]\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + \sigma_z\sigma_x\sigma_z\sin^2\frac{\varphi}{2}\right) \\ &= \frac{\hbar}{2}\left(\sigma_x\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right) - 2\sigma_y\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\right) \\ &= \frac{\hbar}{2}\left(\hat{S}_x\cos\varphi - \hat{S}_y\sin\varphi\right) \end{aligned}$$

for the x-component,

$$\begin{aligned} \hat{S}'_y &= \frac{\hbar}{2}\left(1\cos\frac{\varphi}{2} + i\sigma_z\sin\frac{\varphi}{2}\right)\sigma_y\left(1\cos\frac{\varphi}{2} - i\sigma_z\sin\frac{\varphi}{2}\right) \\ &= \frac{\hbar}{2}\left(\sigma_y\cos^2\frac{\varphi}{2} + i\left[\sigma_z,\sigma_y\right]\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + \sigma_z\sigma_y\sigma_z\sin^2\frac{\varphi}{2}\right) \\ &= \frac{\hbar}{2}\left(\hat{S}_y\cos\varphi + \hat{S}_y\sin\varphi\right) \end{aligned}$$

for the y-component, and, easily,

$$\begin{aligned} \hat{S}'_z &= \frac{\hbar}{2}\left(1\cos\frac{\varphi}{2} + i\sigma_z\sin\frac{\varphi}{2}\right)\sigma_z\left(1\cos\frac{\varphi}{2} - i\sigma_z\sin\frac{\varphi}{2}\right) \\ &= \frac{\hbar}{2}\left(\sigma_z\cos^2\frac{\varphi}{2} - i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + \sigma_z\sin^2\frac{\varphi}{2}\right) \\ &= \hat{S}_z \end{aligned}$$

We have the usual expression for a rotation by φ around the z-axis, which returns to itself after a 2π rotation. Notice that *any* 3-vector would be written as $\mathbf{v}\cdot\boldsymbol{\sigma}$, giving the same result.

3 Neutron interference

Now consider a neutron interference experiment designed to detect this sign difference.

A neutron beam is split into two parallel beams, A and B. Beam B passes through a constant magnetic field $\mathbf{B} = B\mathbf{k}$ for a length l of its path. The beams are then allowed to interfere and the intensity detected. The Hamiltonian is that given above,

$$\begin{aligned}
&= -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} \\
&= -\frac{g_n e}{m_p c} \hat{\mathbf{S}} \cdot \mathbf{B} \\
\hat{H} &= -\frac{g_n e B}{m_p c} \hat{S}_z \\
\hat{H} &= -\omega \hat{S}_z
\end{aligned}$$

We define a frequency,

$$\omega \equiv \frac{g_n e B}{m_p c}$$

The two beams may be represented the states

$$\begin{aligned}
|A\rangle &= e^{-\frac{i}{\hbar}\hat{H}_0 t} |\psi(\mathbf{x}, 0)\rangle (a|+\rangle + b|-\rangle) \\
&= |\psi(\mathbf{x}, t)\rangle (a|+\rangle + b|-\rangle) \\
|B\rangle &= e^{-\frac{i}{\hbar}\hat{H}_0 t - \frac{i}{\hbar}\hat{H}t} |\psi(\mathbf{x}, 0)\rangle (a|+\rangle + b|-\rangle) \\
&= e^{-\frac{i}{\hbar}\hat{H}_0 t} |\psi(\mathbf{x}, 0)\rangle e^{-\frac{i}{\hbar}\hat{H}t} (a|+\rangle + b|-\rangle) \\
&= |\psi(\mathbf{x}, t)\rangle e^{-\frac{i}{\hbar}\hat{H}t} (a|+\rangle + b|-\rangle) \\
&= |\psi(\mathbf{x}, t)\rangle e^{\frac{i\omega t}{\hbar}\hat{S}_z} (a|+\rangle + b|-\rangle) \\
&= |\psi(\mathbf{x}, t)\rangle \left(ae^{\frac{i\omega t}{2}\hat{\sigma}_z}|+\rangle + be^{\frac{i\omega t}{2}\hat{\sigma}_z}|-\rangle\right) \\
&= |\psi(\mathbf{x}, t)\rangle \left(ae^{\frac{i\omega t}{2}}|+\rangle + be^{-\frac{i\omega t}{2}}|-\rangle\right)
\end{aligned}$$

where the free-particle Hamiltonian, $\hat{H}_0 = \frac{\hat{\mathbf{P}}^2}{2m}$, commutes with $\hat{H}$. We take $|A\rangle$ and $|B\rangle$ to be normalized.

Now the beams are recombined. If the beam is traveling in the x-direction, the interference pattern is spread out over the yz plane, and there is a phase difference due to the slightly different distances the beams travel. The combined state

$$\begin{aligned}
|A+B\rangle &= \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle) \\
&= \frac{1}{\sqrt{2}}|\psi(\mathbf{x}, t)\rangle \left((a|+\rangle + b|-\rangle) + \left(ae^{\frac{i\omega t}{2}}|+\rangle + be^{-\frac{i\omega t}{2}}|-\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}|\psi(\mathbf{x}, t)\rangle \left(a\left(1 + e^{\frac{i\omega t}{2}}\right)|+\rangle + b\left(1 + e^{-\frac{i\omega t}{2}}\right)|-\rangle\right)
\end{aligned}$$

with norm

$$\begin{aligned}
\langle A+B | A+B\rangle &= \langle \psi(\mathbf{x}_A, t) + \psi(\mathbf{x}_B, t) | \psi(\mathbf{x}_A, t) + \psi(\mathbf{x}_B, t)\rangle \frac{1}{2}\left(a\left(1 + e^{-\frac{i\omega t}{2}}\right)\langle +| + b\left(1 + e^{\frac{i\omega t}{2}}\right)\langle -|\right)\left(a\left(1 + e^{\frac{i\omega t}{2}}\right)|+\rangle\right. \\
&= \frac{1}{2}f(\mathbf{x}, t)\left(a^2\left(1 + e^{-\frac{i\omega t}{2}}\right)\left(1 + e^{\frac{i\omega t}{2}}\right) + b^2\left(1 + e^{\frac{i\omega t}{2}}\right)\left(1 + e^{-\frac{i\omega t}{2}}\right)\right) \\
&= \frac{1}{2}f(\mathbf{x}, t)\left(a^2\left(2 + 2\cos\frac{\omega t}{2}\right) + b^2\left(2 + 2\cos\frac{\omega t}{2}\right)\right)
\end{aligned}$$

$$\begin{aligned} &= f(\mathbf{x}, t)\left(a^2 + b^2\right)\left(1 + \cos\frac{\omega t}{2}\right) \\ &= f(\mathbf{x}, t)\left(1 + \cos\frac{\omega t}{2}\right) \end{aligned}$$

Therefore, the intensity at a given point oscillates with amplitude proportional to

$$0 \leq 1 + \cos\frac{\omega t}{2} \leq 2$$

with maxima occurring when $\cos\frac{\omega t}{2} = +1$, so the time T between successive maxima satisfies

$$\begin{aligned} \frac{\omega T}{2} &= 2\pi \\ \omega T &= 4\pi \end{aligned}$$

where we see the presence of the 4π rotation. With the velocity of the neutrons related to the reduced deBroglie wavelength, $\bar{\lambda} = \frac{\lambda}{2\pi}$ by

$$\begin{aligned} mv &= \frac{h}{\lambda} \\ v &= \frac{\hbar}{m\bar{\lambda}} \end{aligned}$$

and $T = \frac{l}{v}$, the interference condition becomes

$$\begin{aligned} \omega T &= 4\pi \\ \omega m_n \bar{\lambda} l &= 4\pi\hbar \\ \frac{g_n e B}{m_p c} m_n \bar{\lambda} l &= 4\pi\hbar \\ B &= \frac{4\pi\hbar m_p c}{g_n m_n e \bar{\lambda} l} \end{aligned}$$

or, neglecting the difference between the neutron and proton masses,

$$B = \frac{4\pi\hbar c}{g_n e \bar{\lambda} l}$$