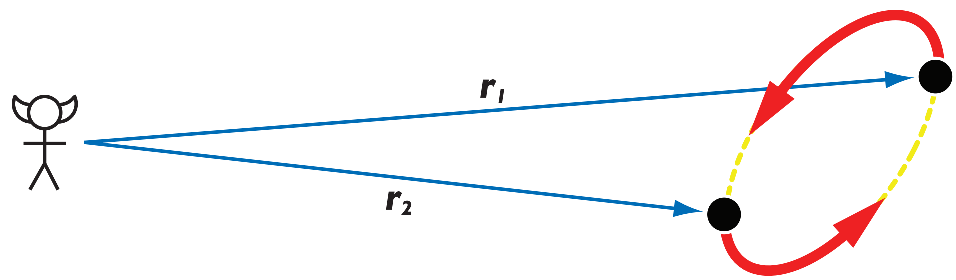
Gravitational Waves

Intuition

• In Newtonian gravity, you can have instantaneous action at a distance. If I suddenly replace the Sun with a $10,000 M_\odot$ black hole, the Earth's orbit should instantly repsond in accordance with Kepler's Third Law. But special relativity forbids this!

• The idea that gravitational information can *propagate* is a consequence of special relativity: nothing can travel faster than the ultimate speed limit, c.



• Imagine observing a distant binary star and trying to measure the gravitational field at your location. It is the sum of the field from the two individual components of the binary, located at distances r_1 and r_2 from you.

• As the binary evolves in its orbit, the masses change their position with respect to you, and so the gravitational field must change. It takes time for that information to propagate from the binary to you — $t_{propagate} = d/c$, where d is the luminosity distance to the binary.

• The propagating effect of that information is known as *gravitational radiation*, which you should think of in analogy with the perhaps more familiar *electromagnetic radiation*

• Far from a source (like the aforementioned binary) we see the gravitational radiation field oscillating and these propagating oscillating disturbances are called *gravitational waves*.

• Like electromagnetic waves

- ▷ Gravitational waves are characterized by a wavelength λ and a frequency f
- ▷ Gravitational waves travel at the speed of light, where $c = \lambda \cdot f$
- ▷ Gravitational waves come in two polarization states (called $+$ [*plus*] and $\times$ [*cross*])

The Metric and the Wave Equation

• There is a long chain of reasoning that leads to the notion of gravitational waves. It begins with the linearization of the field equations, demonstration of gauge transformations in the linearized regime, and the writing of a wave equation for small deviations from the background spacetime. Suffice it to say that this is all eminently well understood and can be derived and proven with a few lectures of diligent work; we will largely avoid this here in favor of illustrating basic results that can be used in applications.

• The traditional approach to the study of gravitational waves makes the assumption that the waves are described by a small perturbation to flat space:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$$

where $\eta_{\mu\nu}$ is the Minkowski metric for flat spacetime, and $h_{\mu\nu}$ is the small perturbations (and often called the *wave metric*). The background metric, $\eta_{\mu\nu}$ is used to raise and lower indices.

• A more general treatment, known as the *Isaacson shortwave approximation*, exists for arbitrary background spacetimes such that

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$$

This approximation works in situations where the perturbative scale of the waves $h_{\mu\nu}$ is much smaller than the curvature scale of the background spacetime $g_{\mu\nu}$. A useful analogy to bear in mind is the surface of an orange — the large scale curvature of the orange (the background spacetime) is much larger than the small scale ripples of the texture on the orange (the small perturbations)

• If one makes the linear approximation above, then the Einstein Equations can be reduced to a vacuum wave equation for the metric perturbation $h^{\mu\nu}$:

$$\Box h^{\mu\nu} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) h^{\mu\nu} = 0 \qquad \rightarrow \qquad \eta^{\alpha\beta} h^{\mu\nu}{}_{,\alpha\beta} = 0$$

• We recognize this is a wave equation, so let's assume that the solutions will be plane waves of the form

$$h^{\mu\nu} = A^{\mu\nu} \exp(i k_\alpha x^\alpha)$$

where $A^{\mu\nu}$ is a tensor with constant components and k_α is a one-form with constant components.

• Taking the first derivative of the solution yields (remember — the components $A^{\mu\nu}$ and k_α are assumed to be constant)

$$h^{\mu\nu}{}_{,\alpha} = k_\alpha h^{\mu\nu}$$

• Taking a second derivative gives us the wave equation back:

$$\eta^{\alpha\beta} h^{\mu\nu}{}_{,\alpha\beta} = \eta^{\alpha\beta} k_\alpha k_\beta h^{\mu\nu} = 0$$

• The only way for this to *generically* be true, is if k_α is null

$$\eta^{\alpha\beta} k_\alpha k_\beta = k_\alpha k^\alpha = 0$$

We call k^α the wave-vector, and it has components $k^\alpha = \{\omega, \vec{k}\}$. The null normalization condition then gives the *dispersion relation*:

$$k_\alpha k^\alpha = 0 \qquad \rightarrow \qquad \omega^2 = k^2$$

• The clean, simple form of the wave-equation noted above has an explicitly chosen gauge condition, called *de Donder gauge* or sometimes *Lorentz gauge* (or sometimes *harmonic gauge*, and sometimes *Hilbert gauge*):

$$h^{\mu\nu}{}_{,\nu} = 0$$

• Since $h^{\mu\nu}$ is symmetric, it in principle has 10 independent coordinates. The choice of this gauge is convenient; it arises in the derivation of the wave equation, and its implementation greatly simplifies the equation (giving the form noted above) by setting many terms to zero. This is very analogous (and should seem familiar to students of electromagnetic theory) to the choice of Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$) in the derivation of the electromagnetic wave equation.

• The choice to use de Donder gauge is part of the gauge freedom we have — the freedom to *choose coordinates*. There are plenty of coordinate systems we could choose to work in, and not have $h^{\mu\nu}{}_{,\nu} = 0$, but the equations would be much more complicated. There is no a priori reason why that should bother us, except it becomes exceedingly difficult to separate *coordinate effects* from *physical effects* (historically, this caused a tremendous amount of confusion for the first 30+ years after Einstein discovered the first wave solutions).

• One can show that choosing de Donder gauge does not use up all the gauge freedom, because small changes in coordinates

$$\bar{x}^\alpha = x^\alpha + \xi^\alpha$$

preserves the gauge if $\xi^{\alpha,\beta}{}_{,\beta} = 0$. This freedom indicates there is still residual gauge freedom, which we can use to simplify the solutions to the wave equation.

• The residual gauge freedom can be used to further constrain the character of $A^{\mu\nu}$. It is desirable to do this, because once all the gauge degrees of freedom are fixed, the remaining independent components of the wave-amplitude $A^{\mu\nu}$ will be physically important. We will skip the derivation, and state the conditions. Using de Donder on our wave solution, we find

$$A^{\mu\nu} k_\nu = 0$$

which tells us that $A^{\mu\nu}$ is *orthogonal* to k^α. We additionally can demand (the gory details are in Schutz, most introductory treatments on gravitational waves; a particularly extensive set of lectures can be found in Schutz & Ricci Lake Como lectures, arxiv:1005.4735):

$$A^\alpha_\alpha = 0$$

and

$$A_{\mu\nu} u^\nu = 0$$

where u^α is a fixed four-velocity of our choice. Together, these three conditions on $A_{\mu\nu}$ are called the *transverse-traceless gauge*.

• What does using all the gauge freedom physically mean? In general relativity, gauge freedom is the freedom to choose coordinates. Here, by restricting the gauge in the wave equation, we are removing the *waving of the coordinates*, which is not a physical effect since coordinates are not physical things (they are human constructs). In essence, if you have a set of particles in your spacetime, the coordinates stay attached to them (this, in and of itself, has no invariant meaning because *you made up the coordinates!*. What is left is the physical effect, the waving of the curvature of spacetime.

• In the transverse-traceless (TT) gauge, there are only 2 independent components of $A_{\mu\nu}$:

$$A^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

• So what is the physical effect of this wave? If we want to build experiments to detect these waves, this question is paramount – we have to know what to look for!

• You might naively look at the geodesic equation and ask what effect the wave has on particle's trajectory, u^α, if that particle is initially at rest (for instance, in the corner of your laboratory). This is an exercise left to the reader, but you will find that given the form of $A_{\mu\nu}$ above, the acceleration of the particle is always zero. If the particle is at rest and never accelerates, it stays at rest!

• This should not surprise us; we said above that the choice of gauge was made to stop the waving of our coordinates! The particle stays at rest because it is attached to the coordinates!

• Experiments should be built around observations that can be used to create *invariant quantities* that all observers agree upon. So rather than a *single* test particle, imagine *two particles* and compute the *proper distance* between them. Imagine both particles begin at rest, one at $x_1^\alpha = \{0,0,0,0\}$ and the other at $x_2^\alpha = \{0,\epsilon,0,0\}$:

$$\ell = \int \sqrt{ds^2} = \int |g_{\alpha\beta} dx^\alpha dx^\beta|^{1/2}$$

Because the particles are separate along the $x-$axis, we integrate along dx and this reduces to

$$\ell = \int_0^\epsilon |g_{xx}|^{1/2} dx \simeq |g_{xx}(x=0)|^{1/2}\epsilon \simeq \left[1 + \frac{1}{2} h^{TT}_{xx}(x=0)\right]\epsilon$$

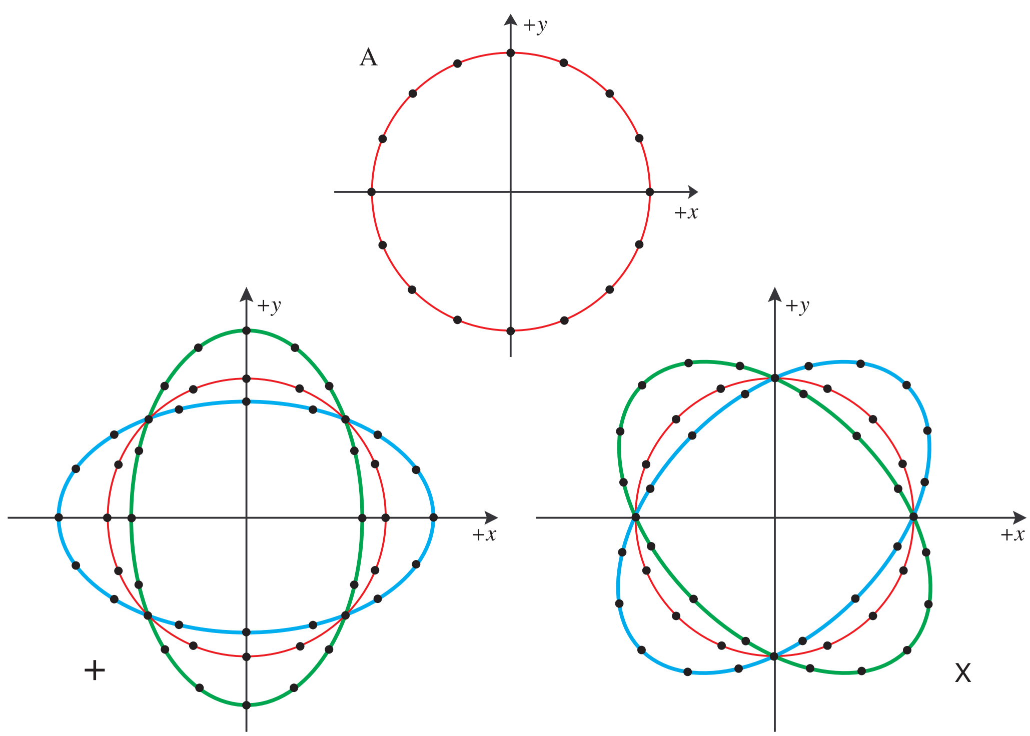
• Now our imposed solution for $h^{TT}_{\mu\nu}$ is a travelling planewave, so h^{TT}_{xx} is not (in general) going to be independent of time. *The proper distance between our test particles changes in time.*

• This is simply *geodesic deviation*, which is the relative trajectories of nearby geodesics in *curved spacetime*. The gravitational wave is curving the spacetime, which we can detect by the geodesic deviation it introduces (gravitational tidal forces).

• This same result can be derived directly from the geodesic deviation equation. It will require you to compute the components of $R^{\alpha}{}_{\beta\gamma\delta}$ in the TT gauge in the presence of $h^{TT}_{\alpha\beta}$.

• Looking at the geodesic deviation by setting first $A_{xx} = 0$ then setting $A_{xy} = 0$ will show that there are two distinct physical states for the wave — these are the gravitational wave *polarization states*. The effect of a wave in either state is to *compress* the geodesics in one direction while simultaneously *stretching* the geodesic separation in the orthogonal direction during the first half-cycle of a wave. During the second half-cycle, it switches the compression and stretching effects between the axes.

• A common way to picture this is to envision a ring of test particles in the $xy-$plane, as shown in A of the figure below. For a gravitational wave propagating up the $z-$axis, choose $A_{xx} \neq 0$ and $A_{xy} = 0$. This will yield the geodesic deviation pattern shown in B of the figure below. The ring initially distorts by stretching along the $y-$axis and compressing along the $x-$axis (the green oval), then a half cycle later compresses and stretches in the reverse directions (the teal oval). This is called the + *(plus) polarization state*. By contrast, $A_{xx} = 0$ and $A_{xy} \neq 0$ produces the distortions shown in C, and is called the × *(cross) polarization state*.



Making Waves: the Quadrupole Formula

• There is an entire industry associated with computing gravitational waveforms, particularly from astrophysical sources.

• Generically, there is a solution to the wave-equation that can be found by integrating over the source, just as there is in electromagnetism. In EM, the vector potential A^μ can be expressed as an integral over the source, the current J^μ. Similarly, in full GR the wave tensor $h_{\mu\nu}$ may be expressed as an integral over the stress-energy tensor $T_{\mu\nu}$:

$$h_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(\vec{x}', t-|\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|} d^3x'$$

• Many sources do not need to be treated fully relativistically. If they are *slow-motion* and the gravitational contribution to the total energy is small, then this expression can be treated in the weak field limit, and reduces to the famous *quadrupole formula*:

$$h^{TT}_{jk} = \frac{2G}{c^4}\frac{1}{r}\ddot{\mathcal{I}}^{TT}_{jk}(t-r/c) \qquad \rightarrow \qquad \frac{2}{r}\ddot{\mathcal{I}}^{TT}_{jk}(t-r)$$

Here $\mathcal{I}_{jk}$ is the *reduced (trace-free) quadrupole moment tensor*, given by

$$\mathcal{I}^{jk} = I^{jk} - \frac{1}{3}\delta^{jk}\delta_{lm}I^{lm}$$

where

$$I^{jk} = \int d^3x\ \rho(t,\vec{x})x^j x^k$$

• The power radiated in gravitational waves (what astronomers call the *luminosity*) is given by

$$\frac{dE_{gw}}{dt} = \frac{G}{c^5}\frac{1}{5}\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}^{jk}\rangle \qquad \rightarrow \qquad \frac{1}{5}\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}^{jk}\rangle$$

Example: Compact Binary System

• In principle the Quadrupole Formula can be used for any system so long as you can compute the components of $\mathcal{I}_{jk}$; in astrophysical scenarios this may require knowledge about the internal mass dynamics of the system that you have no observational access too. Fortunately, astrophysicists are quite fond of models and guessing. :-)

• As an instructive example of the use of the quadrupole formula, consider a circular binary. This is the classic bread and butter source for gravitational wave astronomy. Treating the stars as point masses m_1 and m_2, and confining the orbit to the $xy-$plane, we may write:

$$\begin{aligned} x^i_1 &= r(\theta)\frac{\mu}{m_1}\cdot\{\cos\theta, \sin\theta, 0\} \\ x^i_2 &= r(\theta)\frac{\mu}{m_2}\cdot\{-\cos\theta, -\sin\theta, 0\} \end{aligned}$$

where θ is called the *anomaly* (angular position of the star in its orbit, which changes with time), μ is the reduced mass, defined by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

and $r(\theta)$ is the radius of the orbit as a function of position. Generically, it is defined in terms of the semi-major axis a and the eccentricity e by the *shape equation*:

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$

• For *circular orbits*, the stars are in constant circular motion. You should recall from your General Physics class that in this case the angle θ can be expressed in terms of the angular orbital frequency as

$$\theta = \omega t = 2\pi f_{orb} t = 2\pi \frac{t}{P_{orb}}$$

• We can get a value from ω from Kepler III:

$$GM_T = \omega^2 a^3 \qquad \rightarrow \qquad \omega = \sqrt{\frac{GM_T}{a^3}}$$

In the case of circular orbits, $e = 0$, and so $r(\theta) = a = const$[1]

• Since we are treating the masses as point masses, it is easy to write the mass density ρ in terms of delta-functions:

$$\rho = \delta(z)\left[m_1\delta(x-x_1)\delta(y-y_1) + m_2\delta(x-x_2)\delta(y-y_2)\right]$$

• With these pieces, we can evaluate the components of the quadrupole tensor:

$$\begin{aligned}
I^{xx} &= \int d^3x\left(\rho x^2\right) = m_1 x_1^2 + m_2 x_2^2 \\
&= \left(\frac{\mu^2 a^2}{m_1^2} m_1 + \frac{\mu^2 a^2}{m_2^2} m_2\right)\cos^2(\omega t) \\
&= \mu^2 a^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\cos^2(\omega t) \\
&= \mu a^2 \cos^2(\omega t) \\
&= \frac{1}{2}\mu a^2\left(1+\cos(2\omega t)\right)
\end{aligned}$$

• Notice we have used a trig identity to get rid of the square of the cosine in favor of a term linear in the cosine. The penalty we pay is the frequency of the linear cosine is twice the original orbital frequency.

[1] An astute student will want to compare this with Schutz Eq. 9.94; if one assumes the stars are equal mass, so $M_T = 2m$, and that $a = \ell_o$, one recovers Schutz's result.

• This is a generic feature of circular gravitational wave binaries: *the gravitational wave frequency in a circular binary is twice the orbital frequency.* In practice what it means is that for each cycle made by the binary motion, the gravitational wave signal goes through *two* full cycles — there are two maxima and two minima per orbit. For this reason, gravitational waves are called *quadrupolar waves*.

• Writing out the other components of the quadrupole tensor:

$$I^{yy} = \mu a^2 \sin^2(\omega t) = \frac{1}{2}\mu a^2 \left(1 - \cos(2\omega t)\right)$$

and

$$I^{xy} = I^{yx} = \mu a^2 \cos(\omega t)\sin(\omega t) = \frac{1}{2}\mu a^2 \sin(2\omega t)$$

The trace subtraction is

$$\begin{aligned}\frac{1}{3}\delta^{ij}\delta_{lm}I^{lm} &= \frac{1}{3}\delta^{ij}\mu a^2\left[\frac{1}{2}\left(1+\cos(2\omega t)\right) + \frac{1}{2}\left(1-\cos(2\omega t)\right)\right] \\ &= \frac{1}{3}\delta^{ij}\mu a^2\end{aligned}$$

• These are all the pieces needed to write down the components of $\mathcal{I}^{ij}$

$$\mathcal{I}^{ij} = \frac{1}{2}\mu a^2 \begin{pmatrix} \cos(2\omega t) + 1/3 & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) + 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$$

• Taking two time derivatives of $\mathcal{I}^{ij}$ yields

$$\ddot{\mathcal{I}}^{ij} = 2\mu a^2\omega^2 \begin{pmatrix} -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ -\sin(2\omega t) & \cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

• Taking a third time derivative yields

$$\dddot{\mathcal{I}}^{ij} = 4\mu a^2\omega^3 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

• For circular orbits, these formulae are reasonably easy to work with, especially if you have computer algebra systems like `Maple` or `Mathematica` to help you out. They are somewhat more difficult to work with if the orbits are eccentric.

• For the case of eccentric orbits, the details have been worked out *in extenso* in two papers that have become the de facto starting points for many binary gravitational wave calculations:

- ▶ "Gravitational radiation from point masses in a Keplerian orbit," P. C. Peters and J. Mathews, *Phys. Rev.*, **131**, 435 [1963]
- ▶ "Gravitational radiation from the motion of two point masses," P. C. Peters, *Phys. Rev.*, **136**, 1224 [1964]
- ▶ "The Doppler response to gravitational waves from a binary star source," H. D. Wahlquist, *Gen. Rel. Grav.*, **19**, 1101 [1987]

• The most commonly used results from these papers are as follows. The *average power* (averaged over one period of the elliptical motion) is

$$\langle P \rangle = -\frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 (m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

• In addition to carrying energy away from a binary system, gravitational waves also carry angular momentum. The *angular momentum luminosity* is given by

$$\left\langle \frac{dL}{dt} \right\rangle = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{m_1^2 m_2^2 (m_1+m_2)^{1/2}}{a^{7/2}(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$

• For Keplerian orbits, there are two constants of the motion, generally taken to be the pair $\{E, L\}$, or the pair $\{a, e\}$. The two sets of constants are related, so the luminosities can also be written in terms of the evolution of a and e, written here for completeness:

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{64}{5}\frac{G^3}{c^5}\frac{m_1 m_2 (m_1+m_2)}{a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{304}{15}\frac{G^3}{c^5}\frac{e\, m_1 m_2 (m_1+m_2)}{a^4(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

• If you bleed energy and angular momentum out of an orbit, the masses slowly spiral together until they merge at the center of the orbit! This happens in a finite time called the *coalescence (merger) time*, τ_{merge}. For a circular binary with initial semi-major axis a_o, the expression for $\langle da/dt \rangle$ can be integrated to give

$$\tau_{circ}(a_o) = \frac{a_o^4}{4\beta}$$

where the constant β is defined as

$$\beta = \frac{64}{5}\frac{G^3}{c^5} m_1 m_2 (m_1+m_2)$$

• For a general binary with initial parameters $\{a_o, e_o\}$ it is given by

$$\tau_{merge}(a_o, e_o) = \frac{12}{19}\frac{c_o^4}{\beta}\int_0^{e_o} de \frac{e^{29/19}\left[1 + (121/304)e^2\right]^{1181/2299}}{(1-e^2)^{3/2}}$$

where the constant c_o is given by

$$c_o = \frac{a_o(1-e_o^2)}{e_o^{12/19}}\left[1 + \frac{121}{304}e_o^2\right]^{-870/2299}$$

• It is often useful to consider limiting cases. For e_o small, we should get a lifetime similar to τ_{circ}. Expanding the lifetime for small e_o yields

$$\tau_{merge}(a_o, e_o) \simeq \frac{12}{19}\frac{c_o^4}{\beta}\int_0^{e_o} de\, e^{29/19} = \frac{c_o^4}{4\beta}e_o^{48/19}$$

This is approximately equal to $\tau_{circ}(a_o)$.

• For e_o near 1 (a marginally bound orbit that will evolve through emission of gravitational radiation — this is often called a *capture orbit*)

$$\tau_{merge}(a_o, e_o) \simeq \frac{768}{425}\tau_{circ}(a_o)(1-e_o^2)^{7/2}$$

Pocket Formulae for Gravitational Wave Binaries

• Because binaries are expected to be among the most prevalent of gravitational wave sources, it is useful to have a set of pocket formulae for quickly estimating their characteristics on the back of old cell phone bills; you can go back and do all the crazy stuff above if you need an accurate computation.

• For a gravitational wave binary with masses m_1 and m_2, in a circular orbit with gravitational wave frequency $f = 2f_{orb}$, then:

chirp mass $$\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

scaling amplitude $$h_o = 4\frac{G}{c^2}\frac{\mathcal{M}_c}{D}\left(\frac{G}{c^3}\pi f \mathcal{M}_c\right)^{2/3}$$

chirp $$\dot{f} = \frac{96}{5}\frac{c^3}{G}\frac{f}{\mathcal{M}_c}\left(\frac{G}{c^3}\pi f \mathcal{M}_c\right)^{8/3}$$

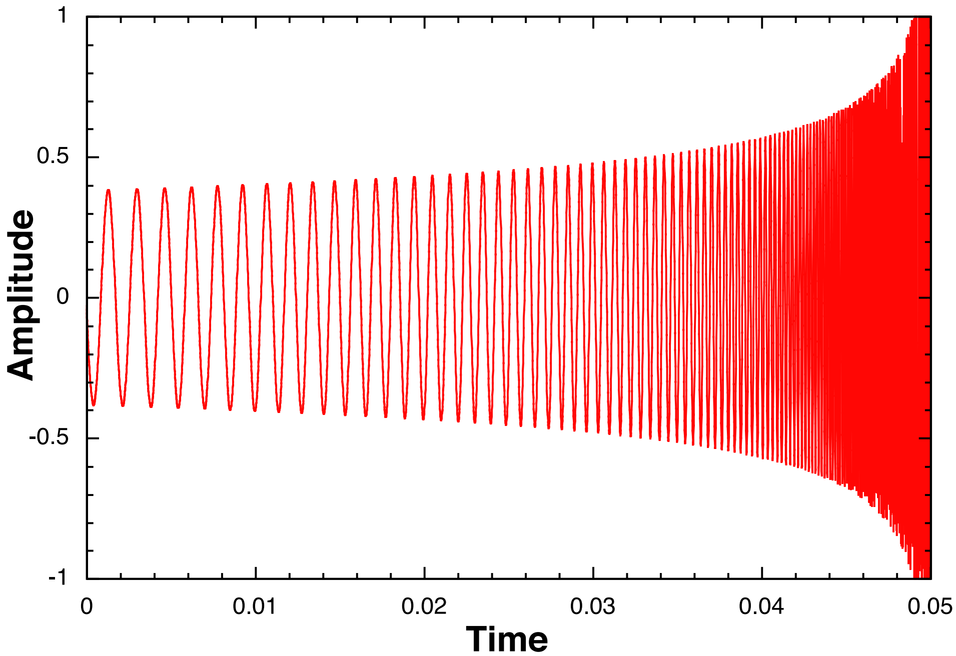
• The *chirp* indicates that as gravitational waves are emitted, they carry energy away from the binary. The gravitational binding energy decreases, and the orbital frequency increases. The gravitational wave *phase* $\phi(t)$ evolves in time as

$$\phi(t) = 2\pi\left(f\ t + \frac{1}{2}\dot{f}\ t^2\right) + \phi_o\ ,$$

where $\dot{f}$ is the chirp given above, and ϕ_o is the initial phase of the binary. A phenomenological form of the waveform then is given by

$$h(t) = h_o \cos\phi(t) = h_o \cos\left(2\pi f\ t + \pi \dot{f}\ t^2 + \phi_o\right)$$

• This expression has all the qualitative properties of a coalescing waveform, shown below.



• This is called a *chirp* or a *chirp waveform*, characterized by an increase in amplitude and frequency as time increases. This name is quite suitable because of the way it sounds if the amplitude is increased by a large factor and the waveform is dumped into an audio generator.

Luminosity Distance from Chirping Binaries

Suppose I can measure the chirp $\dot{f}$ and the gravitational wave amplitude h_o. The chirp can be inverted to give the chirp mass:

$$\mathcal{M}_c = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f^{-11/3} \dot{f}\right]^{3/5}$$

If this chirp mass is used in the amplitude equation, one can solve for the *luminosity distance* D:

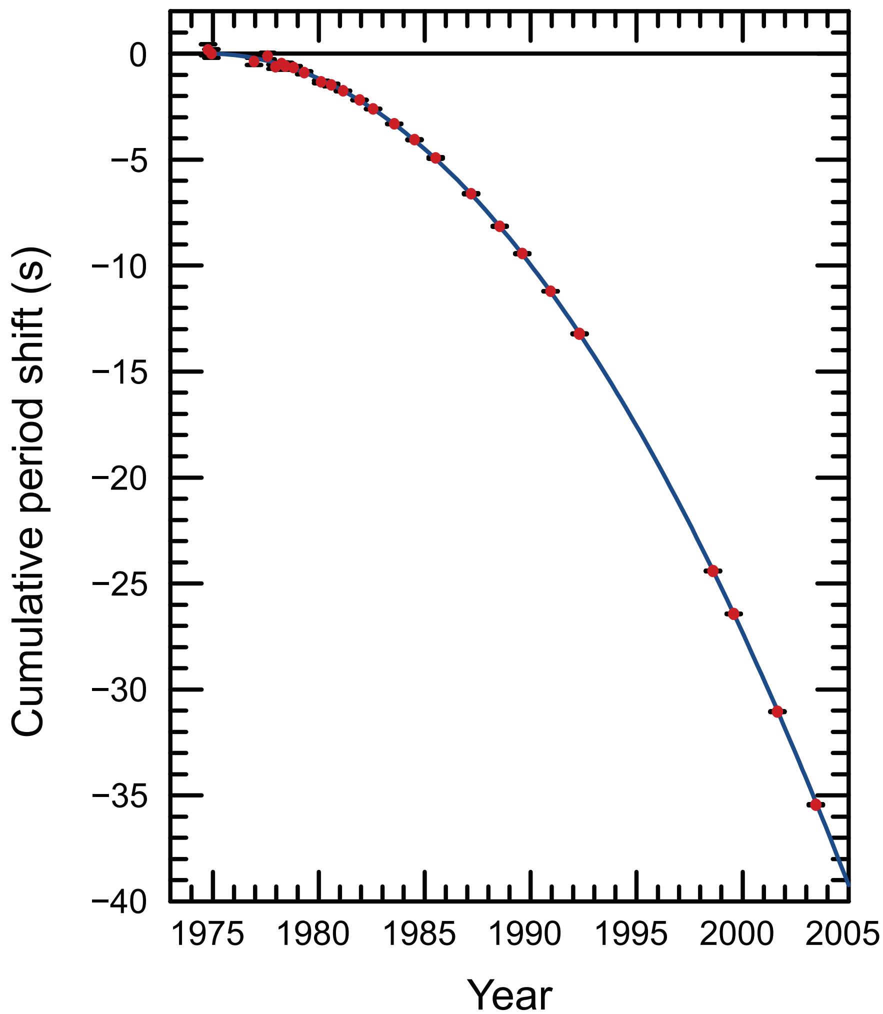
$$D = \frac{5}{96\pi^2}\frac{c}{h_o}\frac{\dot{f}}{f^3}$$

This is a method of measuring the luminosity distance using *only gravitational wave observables*! This is extremely useful as an independent distance indicator in astronomy.

Application: Binary Pulsar

• Early on we became confident in the existence of gravitational waves because we could observe their astrophysical influence. The first case of this was the pulsar, PSR $B1913+16$, my colloquially known as "The Binary Pulsar," or the "Hulse-Taylor Binary Pulsar," after the two radio astronomers who discovered it in 1974.

bullet The Binary Pulsar is famous because it is slowly spiraling together. As shown in the figure below, the rate at which the binary is losing energy from its orbit is *precisely* what is expected from general relativity! This is the strongest, *indirect* observational evidence for the existence of gravitational waves. Joe Taylor and Russell Hulse received the Nobel Prize for this discovery in 1993.



• Let's use our formulae for inspiralling binaries to examine the binary pulsar in detail. The physical parameters of this system are given in the table below.

Symbol	***Name***	***Value***
m_1	primary mass	$1.441 M_\odot$
m_2	secondary mass	$1.387 M_\odot$
P_{orb}	orbital period	7.751939106 hr
a	semi-major axis	1.9501×10^9 m
e	eccentricity	0.617131
D	distance	$21,000$ lyr

• If one computes the yearly change in semi-major axis, one finds

$$\left\langle \frac{da}{dt} \right\rangle = 3.5259 \ \frac{m}{yr}$$

which is *precisely* the measured value from radio astronomy observations!

• Because gravitational waves are slowly bleeding energy and angular momentum out of the system, the two neutron stars will one day come into contact, and *coalesce* into a single, compact remnant. The time for that to happen is

$$\tau_{merge} = 3.02 \times 10^8 \ yr$$

• This is well outside the lifetime of the average astronomer, and longer than the entire history of observational astronomy on the planet Earth! It is, however, much shorter than a Hubble time! This suggests the since (a) there are many binary systems in the galaxy, and (b) neutron stars are not an uncommon end state for massive stars to evolve to, then there should be *many* binary neutron stars coalescing in the Universe as a function of time.

• This is the first inkling we have that there could be many such sources in the sky, and that perhaps observing them in gravitational waves could be a useful observational exercise.

• If we are going to contemplate observing then, we should have some inkling of their strength. What is the scaling amplitude, h_o of the Hulse-Taylor binary pulsar?

$$h_o = 4.5 \times 10^{-23}$$

This number is extremely small, but we haven't talked about whether it is detectable or not. Let's examine this in the context of building a detector.

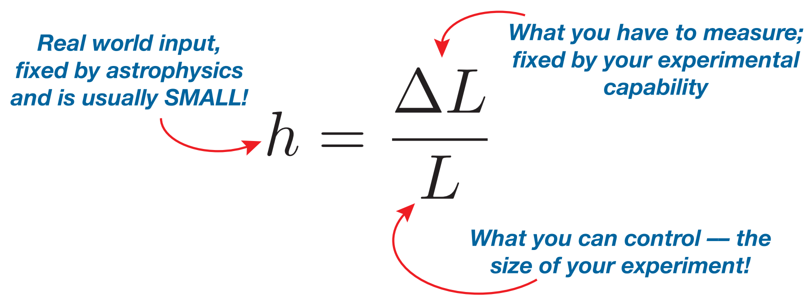
Detector Sensitivity

• When you decide to build a detector, you think about the physical effect you have to measure. We have seen that gravitational waves *change the proper distance between particles.*

We characterize this distance by the *strain* $h = \Delta L/L$. This fundamental definition guides our basic thinking about detector design. If ΔL is what we have to measure, over the distance L, then the kind of astrophysical strain from typical astrophysical objects is roughly

$$h = \frac{\Delta L}{L} \sim 10^{-21} \sim \frac{\text{Diameter of H atom}}{1 \text{ AU}}$$

• The way these quantities enter in the process of experiment design is shown schematically below:



• There are two ways to go about this. You could decide what astrophysical sources you are interested in, and determine what detector is needed, or you can decide what detector you can build (L is determined by size and pocketbook, whereas ΔL is fixed by the ingenuity of your experimentalists). But often the design problem is an optimization of *both* astrophysics and capability.

• In the modern era, gravitational wave detection technology is dominated by *laser interferometers*, which we will focus on here. In general, an interferometric observatory has its best response at the *transfer frequency* $f_\star$, where gravitational wavelengths are roughly the distance probed by the time of flight of the lasers:

$$f_\star = \frac{c}{2\pi L}$$

• If you build a detector, the principle goal is to determine what gravitational waves the instrument will be sensitive to. We characterize the noise in the instrument and the instrument's response to gravitational waves using a *sensitivity curve*.

• Sensitivity curves plot the strength a source must have, as a function of gravitational wave frequency, to be detectable. There are two standard curves used by the community:

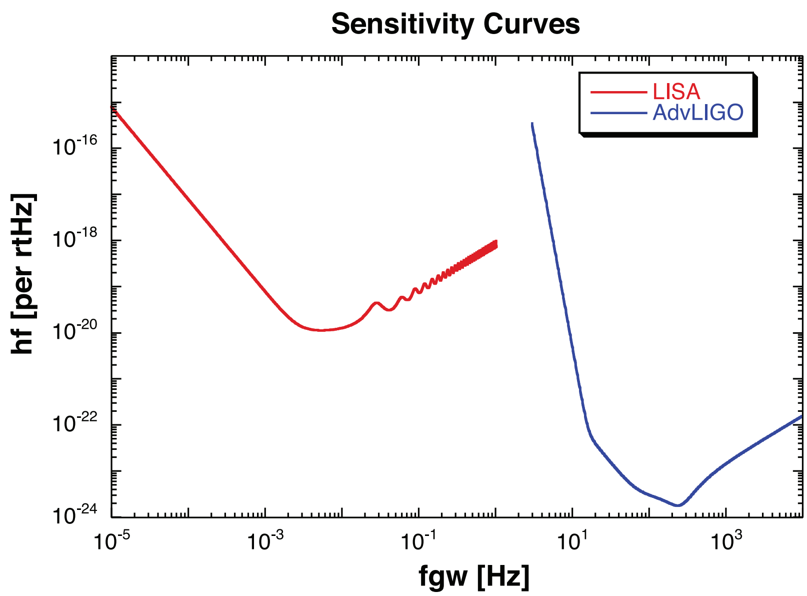
▷ **Strain Sensitivity.** This plots the gravitational wave strain amplitude h versus gravitational wave frequency f.

▷ **Strain Spectral Amplitude.** This plots the square root of the power spectral density, $h_f = \sqrt{S_h}$ versus gravitational wave frequency f. The power spectral density is the power per unit frequency and is often a more desirable quantity to work with because gravitational wave sources often evolve dramatically in frequency during observations.

• The strain sensitivity of a detector, h^D, builds up over time. If you know the observation time T_{obs} and the spectral amplitude curve (like those plotted above) you can convert between the two via

$$h_f^D = h^D \sqrt{T_{obs}}$$

• The sensitivity for LIGO and LISA are shown below. Your own LISA curves can be created using the online tool at `www.srl.caltech.edu/~shane/sensitivity/MakeCurve.html`.



• LISA has armlengths of $L = 5 \times 10^9$ m, which if you consider its transfer frequency $f_\star$ makes it more sensitive at lower frequencies. LIGO has armlengths of $L = 4$ km, but the arms are Fabry-Perot cavities, and the laser light bounces back and forth ~ 100 times; this puts its prime sensitivity at a much higher frequency.

Sources and Sensitivity Curves

• Sensitivity curves are used to determine whether or not a source is detectable. Rudimentarily, if the strength of the source places it above the sensitivity curve, it can be detected! How do I plot sources on these curves? First, it depends on what kind of curve you are looking at; second, it depends on what kind of source you are working with!

• If you are talking about observing sources that are evolving slowly (the are approximately monochromatic) then the spectral amplitude and strain are related by

$$h_f = h\sqrt{T_{obs}}$$

• If you are talking about a short-lived ("bursting") source with a characteristic width τ, then to a good approximation the bandwidth of the source in frequency space is $\Delta f \sim \tau^{-1}$

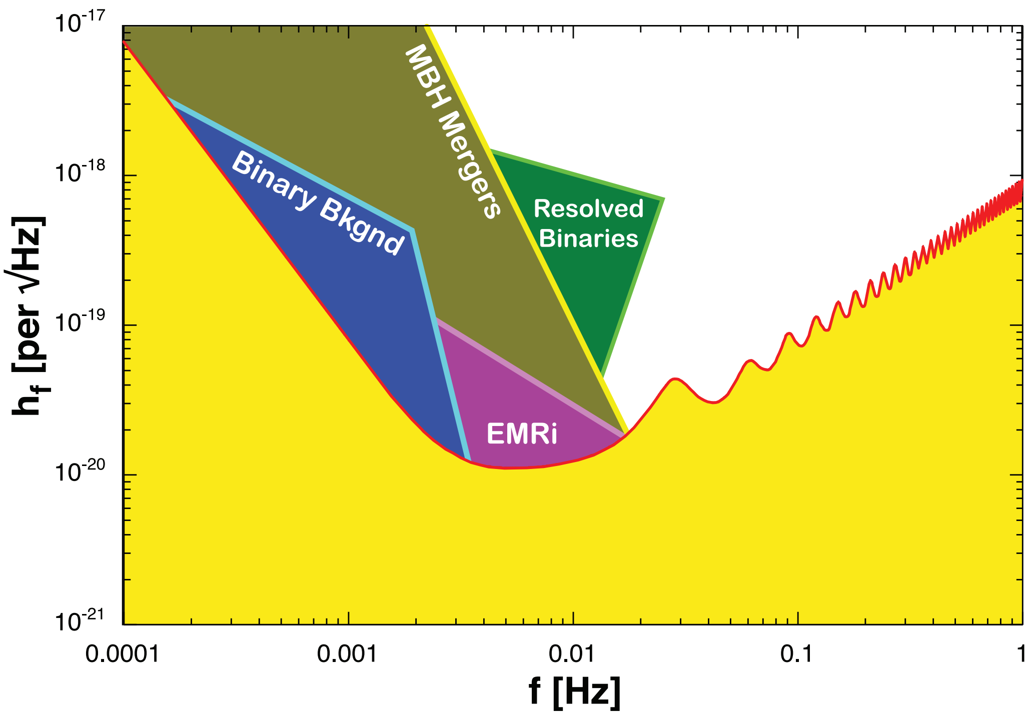
and the spectral amplitude and strain are related by

$$h_f = \frac{h}{\sqrt{\Delta f}} = h\sqrt{\tau}$$

• The fundamental metric for detection is the SNR ρ (signal to noise ratio) defined as

$$\rho \sim \frac{h_f^{src}}{h_f^D}$$

• To use this you need to know how to compute h_f^{src}. A good starting point is the pocket formulae from the last section.



Rosetta Stone: Orbital Mumbo Jumbo ▸ . . . .

- **a = semi-major axis.** The *major axis* is the long axis of the ellipse. The semi-major axis is 1/2 this length.
- **b = semi-minor axis.** The *minor axis* is the short axis of the ellipse. The semi-minor axis is 1/2 this length.
- **e = eccentricity.** The eccentricity characterizes the deviation of the ellipse from circular; when $e = 0$ the ellipse is a circle, and when $e = 1$ the ellipse is a parabola. The eccentricity is defined in terms of the semi-major and semi-minor axes as

$$e = \sqrt{1 - (b/a)^2}$$

- **f = focus.** The distance from the geometric center of the ellipse (where the semi-major and semi-minor axes cross) to either focus is

$$f = ae$$

- ℓ = **semi-latus rectum.** The distance from the focus to the ellipse, measured along a line parallel to the semi-minor axis, and has length

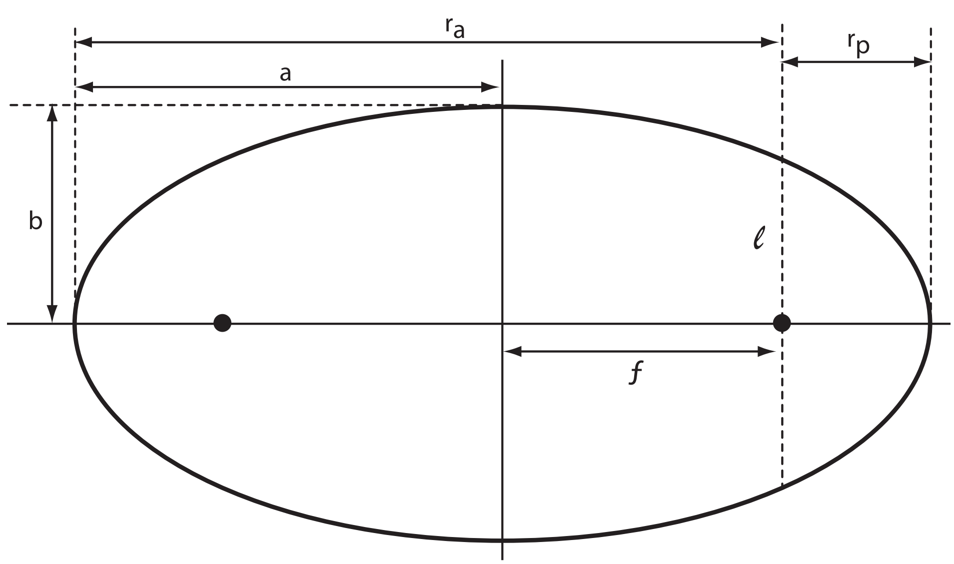
$$\ell = b^2/a$$

- r_p = **periapsis.** The periapsis is the distance from the focus to the nearest point of approach of the ellipse; this will be along the semi-major axis and is equal to

$$r_p = a(1 - e)$$

- r_a = **apoapsis.** The apoapsis is the distance from the focus to the farthest point of approach of the ellipse; this will be along the semi-major axis and is equal to

$$r_a = a(1 + e)$$

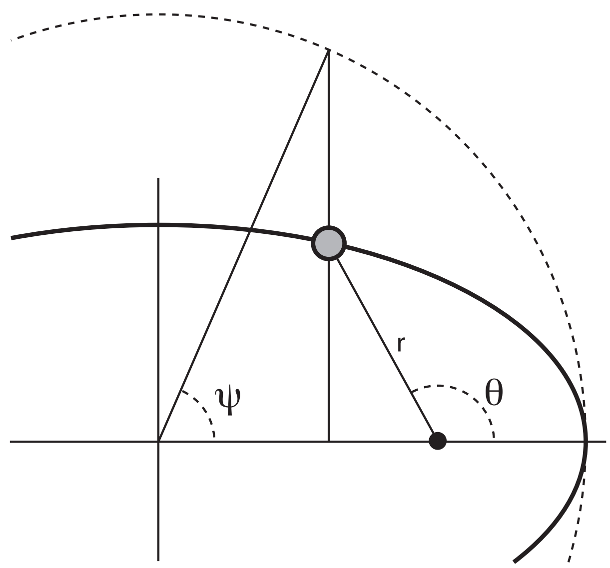


Basic Geometric Definitions ▶ . . . .
The game of orbits is always about locating the positions of the masses. For planar orbits (the usual situation we encounter in most astrophysical applications) one can think of the position of the mass m_i in terms of the Cartesian coordinates $\{x_i, y_i\}$, or in terms of some polar coordinates $\{r_i, \theta_i\}$. The value of the components of these location vectors generically depends on the coordinates used to describe them. The most common coordinates used are called *barycentric coordinates*, with the origin located at the focus between the two bodies.

▷ **The Shape Equation**. The shape equation gives the distance of the orbiting body (*"particle"*) from the focus of the orbit as a function of polar angle θ. It can be expressed in various ways depending on the parameters you find most convenient to describe the orbit.

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad \to \qquad r = \frac{r_p(1+e)}{1+e\cos\theta} \qquad \to \qquad r = \frac{r_a(1-e)}{1+e\cos\theta}$$

▷ **The Anomaly**. Astronomers refer to the angular position of the body as the *anomaly*. There are three different anomalies of interest.



• θ = **true anomaly.** This is the polar angle θ measured in barycentric coordinates.

• $\mathcal{M}$ = **mean anomaly.** This is the phase of the orbit expressed in terms of the time t since the particle last passed a reference point, generally taken to be $\theta = 0$

$$\mathcal{M} = \frac{2\pi}{P} t$$

Note that for *circular orbits*, $\theta = \mathcal{M}$.

• ψ = **eccentric anomaly.** This is a geometrically defined angle measured from the center of the ellipse to a point on a circumferential circle with radius equal to the semi-major axis of the ellipse. The point on the circle is geometrically located by drawing a perpendicular line from the semi-major axis of the ellipse through the location of the particle. The eccentric anomaly is important for locating the position of the particle as a function of time (using a construction known as *the Kepler Equation*, not to be confused with the three laws of orbital motion).