Magnetic forces and the magnetic field

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Here we develop the Lorentz force law, and the Biot-Savart law for the magnetic field.

1 The Lorentz force law

Like electrostatics, magnetostatics begins with the force on a charged particle.

1.1 The force law

A beam of electrons passing a permanent magnet is deflected in a direction perpendicular to the velocity. Moving electrons – a current in a wire – deflects a compass needle brought nearby. These observations show that electric and magnetic phenomena influence one another.

Careful measurement shows that in the presence of both electric and magnetic fields, the force on a charge Q is

$$\mathbf{F} = Q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

This is the *Lorentz force law*. Notice that this is consistent with the deflection described above since the cross product $\mathbf{v} \times \mathbf{B}$ is always perpendicular to both $\mathbf{v}$ and $\mathbf{B}$.

Example: Motion of a particle in a constant magnetic field A particle of charge Q with initial velocity $\mathbf{v}_0$ moves in a constant magnetic field of magnitude B_0. Find the motion.

Choose the z-axis along the direction of the constant field, so that $\mathbf{B} = B_0\hat{\mathbf{k}}$, and choose the x-direction so that the initial velocity is $\mathbf{v}_0 = v_{0x}\hat{\mathbf{i}} + v_{0z}\hat{\mathbf{k}}$. The Lorentz law gives the force. Substituting the magnetic force into Newton's second law, gives

$$Q\mathbf{v} \times \mathbf{B} = m\frac{d\mathbf{v}}{dt}$$

Writing out the separate components,

$$\begin{aligned} Q\left(v_x B_y - v_y B_x\right) &= m\frac{dv_z}{dt} \\ Q\left(v_y B_z - v_z B_y\right) &= m\frac{dv_x}{dt} \\ Q\left(v_z B_x - v_x B_z\right) &= m\frac{dv_y}{dt} \end{aligned}$$

Dropping $B_x = B_y = 0$, we have

$$\begin{aligned} \frac{dv_z}{dt} &= 0 \\ Qv_y B_z &= m\frac{dv_x}{dt} \\ -Qv_x B_z &= m\frac{dv_y}{dt} \end{aligned}$$

The first equation immediately integrates twice to give

$$z = z_0 + v_{0z} t$$

so the charge moves with constant velocity in the direction parallel to the field. For the remaining two equations, differentiate each with respect to t,

$$\begin{aligned} Q\frac{dv_y}{dt}B_z &= m\frac{d^2 v_x}{dt^2} \\ -Q\frac{dv_x}{dt}B_z &= m\frac{d^2 v_y}{dt^2} \end{aligned}$$

and substitute the original equations to eliminate the first derivative terms and separate v_x and v_y,

$$\begin{aligned} \frac{d^2 v_x}{dt^2} + \frac{Q^2 B_z^2}{m^2} v_x &= 0 \\ \frac{d^2 v_y}{dt^2} + \frac{Q^2 B_z^2}{m^2} v_y &= 0 \end{aligned}$$

The solutions are clearly harmonic. Define the *cyclotron frequency*,

$$\omega \equiv \frac{QB_z}{m}$$

so that

$$\begin{aligned} v_x &= A \sin \omega t + B \cos \omega t \\ v_y &= C \sin \omega t + D \cos \omega t \end{aligned}$$

These must satisfy the original equations

$$\begin{aligned} \frac{dv_x}{dt} &= \omega v_y \\ \frac{dv_y}{dt} &= -\omega v_x \end{aligned}$$

with the given initial conditions. Therefore, we must have

$$\begin{aligned} \omega A &= \omega D \\ -\omega B &= \omega C \\ \omega C &= -\omega B \\ -\omega D &= -\omega A \end{aligned}$$

so that

$$\begin{aligned} v_x &= A \sin \omega t + B \cos \omega t \\ v_y &= -B \sin \omega t + A \cos \omega t \end{aligned}$$

Fitting the initial values, we see that $v_{x0} = B$ and $0 = v_{0y} = A$. Therefore, the velocity in the xy-plane is a circle given by

$$\begin{aligned} v_x &= v_{0x} \cos \omega t \\ v_y &= -v_{0x} \sin \omega t \end{aligned}$$

The full motion is a spiral parallel to the z-axis.

1.1.1 Force on a current in a wire

Next, we consider how a current is affected by a magnetic field. The current is comprised of moving charges. If the charges move at speed $v = \frac{dl}{dt}$, then in a time dt all charges in the interval $d\mathbf{l} = \mathbf{v}dt$ pass a given point. If the number of moving charges per unit length of the wire is $\lambda = \frac{dq}{dl}$, then the amount of charge passing that point in time dt is

$$dq = \lambda v dt$$

so the magnitude of the current is $I = \frac{dq}{dt} = \lambda v$.

Now consider an element of charge, $dq = \lambda dl$ moving with speed v in a magnetic field $\mathbf{B}$. The Lorentz force law gives the force on dq,

$$\begin{aligned} d\mathbf{F} &= dq\mathbf{v} \times \mathbf{B} \\ &= \frac{dq}{dt} (\mathbf{v} \times \mathbf{B})\, dt \\ &= I\,(d\mathbf{l} \times \mathbf{B}) \end{aligned}$$

Replacing the velocity using $\mathbf{v} = \frac{d\mathbf{l}}{dt}$ for a displacement $d\mathbf{l}$ along the wire, this becomes

$$d\mathbf{F} = I\,(d\mathbf{l} \times \mathbf{B})$$

Integrating along the length of the wire we get the total force on a current carrying wire,

$$\begin{aligned} \mathbf{F} &= \int I\,(d\mathbf{l} \times \mathbf{B}) \\ &= I \int d\mathbf{l} \times \mathbf{B} \end{aligned}$$

Example: Breaking a wire Consider a circular loop of wire in a magnetic field of radius $.1\,m$. The steel wire has a tensile strength of $400\,MPa$, and a cross-sectional radius of $.25\,mm$. How strong must a magnetic field through the loop be in order to break the wire if the wire carries a current of $10\ A$?

The tensile strength of $400\ MPa$ is

$$4 \times 10^8 \frac{N}{m^2}$$

while the cross-sectional area of the wire is

$$\pi r^2 = \pi \times \left(.25 \times 10^{-3}\right)^2 = .196 \times 10^{-6} m^2$$

so the magnitude of the force required to break the wire is

$$F = 4 \times 10^8 \frac{N}{m^2} \times .196 \times 10^{-6} m^2 = 78.5\,N$$

Assuming the magnetic field is perpendicular to the plane of the loop, the outward magnetic force on an infinitesimal length dl of the wire is

$$F = IBdl$$

Now this force must be offset by the tension in the wire. If dl subtends an angle $d\theta$ then $dl = Rd\theta$, while the tensions at the ends of the segment, being tangent to the circle at their respective locations, aim at angles differing by $d\theta$. The inward component of each tension is

$$T_\perp = T \sin\frac{d\theta}{2} \approx \frac{1}{2} T d\theta$$

so the total inward force due to tension is

$$F_{inward} = 2T_{\perp} \approx Td\theta$$

As long as the wire is in equilibrium, the inward and outward forces must be equal:

$$\begin{aligned} F_{out} &= F_{in} \\ IBRd\theta &= Td\theta \\ IBR &= T \end{aligned}$$

The magnetic field required to break the wire is therefore

$$\begin{aligned} B &= \frac{T}{IR} \\ &= \frac{78.5\,N}{10\,A \times .1\,m} \\ B &= 78.5\,Tesla \end{aligned}$$

This calculation neglects the field produced by the current in the wire.

Currently, the strongest (pulsed) magnetic field yet obtained non-destructively in a laboratory (National High Magnetic Field Laboratory, LANL) as of 2012 is 100.75 Tesla.

2 Current density

It turns out to be more appropriate to treat the current as a current density vector rather than a simple current scalar, I, since the current may vary in both direction and magnitude from place to place. The current density captures both of these features.

A current I may be viewed as made up of many charges in a (microscopically large, macroscopically small) region d^3x moving with velocity $\mathbf{v}(\mathbf{x})$. If the density of charges at $\mathbf{x}$ is ρ, then there is a current density, $\mathbf{J} = \rho\mathbf{v}$. The current vector $\mathbf{I}$ is then the integral of a current density in a region.

We have defined $\mathbf{J}$ so that the amount of charge in volume d^3x moving with velocity v in the direction of $\mathbf{J}$ is given by

$$dq = \frac{\mathbf{J}d^3x}{v}$$

If we write the volume element in terms of the displacement $dl = vdt$ due to the motion and and area element d^2x orthogonal to this,

$$d^3x = dld^2x$$

then

$$\begin{aligned} dq &= \frac{\mathbf{J}d^3x}{v} \\ &= \frac{\mathbf{J}dld^2x}{dl/dt} \\ &= \mathbf{J}d^2xdt \end{aligned}$$

Thus, we think of $\mathbf{J}$ as the amount of charge crossing the surface d^2x in time dt.

Dividing this expression by dt and integrating over a cross-section of the current-carrying wire, the current crossing any surface S is

$$I = \frac{dq}{dt} = \iint_S \mathbf{J} \cdot \hat{\mathbf{s}} d^2x$$

where $\hat{\mathbf{s}}$ is the unit normal to S in the direction of the current.

2.1 Conservation of charge

Suppose we have a region of space with charge density ρ. Let some or all of this charge move as a current density, $\mathbf{J}$. Now, since we find that total charge is conserved, we know that the total charge in some volume $\mathcal{V}$ can only change if the current carries charge across the boundary $\mathcal{S}$ if $\mathcal{V}$. Therefore, with the charge in the volume $\mathcal{V}$ given by

$$Q_{tot} = \int_{\mathcal{V}} \rho d^3x$$

the time rate of change of Q_{tot} must be given by the total flux $\mathbf{J}$ across the boundary. Let $\hat{\mathbf{n}}$ be the outward normal of the boundary $\mathcal{S}$ of $\mathcal{V}$. Then

$$\frac{dQ_{tot}}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}} d^2x \tag{1}$$

On the left side, we rewrite $\frac{dQ_{tot}}{dt}$ by interchanging the order of integration and differentiation ,

$$\begin{aligned} \frac{dQ_{tot}}{dt} &= \frac{d}{dt}\int_{\mathcal{V}} \rho d^3x \\ &= \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d^3x \end{aligned}$$

while on the right we use the divergence theorem, $-\oint_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}} d^2x = -\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{J} d^3x$. Substituting both these changes into eq.(1), we have

$$\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} \right) d^3x = 0$$

Since the final equation holds for all volumes $\mathcal{V}$ it must hold at each point, leading us to the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{2}$$

An equation of this sort holds anytime there is a conserved quantity.

We define a *steady state current* to be one for which ρ and $\mathbf{J}$ are independent of explicit time dependence, $\frac{\partial \rho}{\partial t} = 0$, $\frac{\partial \mathbf{J}}{\partial t} = 0$. For a steady state current, the current density has vanishing divergence, $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$.

2.2 Examples of current density

a) A wire of rectangular cross section with sides a and b carries a total current I. Find the current density if the flow is uniformly distributed through the wire.

The current density is defined so

$$I = \frac{dq}{dt} = \iint \mathbf{J} \cdot \hat{\mathbf{s}} d^2x$$

Taking the z-direction for the direction of the current, and noting that $\mathbf{J}$ is constant across the cross-section, we have

$$\begin{aligned} I\hat{\mathbf{k}} &= J\hat{\mathbf{k}} \int_0^a dx \int_0^b dy \\ I &= Jab \end{aligned}$$

and therefore

$$\mathbf{J} = \frac{I}{ab}\hat{\mathbf{k}}$$

If we want $\mathbf{J}$ to be defined everywhere, we may use theta functions to limit the range,

$$\mathbf{J} = \frac{I}{ab}\hat{\mathbf{k}}\Theta(a-x)\Theta(x)\Theta(b-y)\Theta(y)$$

b) A thick disk of radius R and thickness L has a uniform charge density throughout its volume so that the total charge is Q. The disk rotates with angular velocity ω. What is the current density inside the disk? What is the current flowing between radius r and $r+dr$? What is the total current?

First, the charge density in the disk is the total charge over the total volume of the disk,

$$\rho = \frac{Q}{\pi R^2 L}$$

The velocity at radius r is $\omega r\hat{\boldsymbol{\varphi}}$, so

$$\mathbf{J} = \rho\mathbf{v} = \frac{Qr\omega}{\pi R^2 L}\hat{\boldsymbol{\varphi}}$$

The current at radius r through a cross-section of width dr and height L is the integral

$$\begin{aligned} dI &= \int_0^L \frac{Qr\omega}{\pi R^2 L} dr dz \hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}} \\ &= \frac{Q\omega r dr}{\pi R^2} \end{aligned}$$

The total current follows if we integrate across the radius of the cross-section,

$$\begin{aligned} I &= \int_0^R dI \\ &= \int_0^R \frac{Q\omega r dr}{\pi R^2} \\ &= \frac{Q\omega R^2}{2\pi R^2} \\ &= \frac{Q\omega}{2\pi} \end{aligned}$$

This makes sense, since $\frac{\omega}{2\pi}$ is the fraction of a circle per unit time that the disk rotates, so we have that fraction of the total charge passing per unit time.

3 The Biot-Savart law

3.1 The magnetic field due to a given current density

Next, we consider the effect of a current in *producing* a magnetic field. The experimental results are summarized for steady state line currents by the *Biot-Savart law*,

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x})\times(\mathbf{x}_0-\mathbf{x})}{|\mathbf{x}_0-\mathbf{x}|^3} d^3x$$

The constant μ_0 is the *permeability of free space* with the value $\mu_0 = 4\pi \times 10^{-7} N/A^2$. There is a clear parallel with our equation for the electric field. Instead of simply the charge density times the factor $\frac{\mathbf{x}_0-\mathbf{x}}{|\mathbf{x}_0-\mathbf{x}|^3}$, we now require the cross product with the current density.

We can simplify this expression in the case of a current carrying wire. Let the current density be restricted to a wire in the x-direction so that

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{i}} I \delta(y) \delta(z)$$

Then

$$\begin{aligned} \mathbf{B}(\mathbf{x}_0) &= \frac{\mu_0}{4\pi} \int \frac{\hat{\mathbf{i}} I \delta(y) \delta(z) \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} d^3x \\ &= \frac{\mu_0}{4\pi} \int \frac{I \left(\hat{\mathbf{i}} dx\right) \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} \end{aligned}$$

This is now integrated along the length of the wire. More generally, let the current be in the direction $d\mathbf{l}$. Then we have the Biot-Savart law for a current,

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l} \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}$$

where the currents move along wires with tangents $d\mathbf{l}$ at positions $\mathbf{x}$. This was the original form of the law.

The law may also be specialized to surface current density $\mathbf{K}(\mathbf{x})$ by simply restricting the current density to a surface:

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{x}) \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3} d^2x$$

Now that we have an expression for the magnetic field in terms of source current density, we can find the laws of magnetostatics.

3.2 Magnetic field of a long straight wire

Find the magnetic field above a long straight wire carrying a steady current I.

Using the Biot-Savart law for a steady current, we have

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l} \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}$$

Using cylindrical coordinates, let the current flow in the z-direction, so that $d\mathbf{l} = dz\hat{\mathbf{k}}$. Then $\mathbf{x} = z\hat{\mathbf{k}}$ and we take $\mathbf{x}_0 = \rho\hat{\boldsymbol{\rho}}$ so that the observation point is a distance ρ radially out from the wire. Evaluating the integrand, we have

$$\begin{aligned} \mathbf{B}(\mathbf{x}_0) &= \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{I dz \hat{\mathbf{k}} \times \left(\rho\hat{\boldsymbol{\rho}} - z\hat{\mathbf{k}}\right)}{\left|\rho\hat{\boldsymbol{\rho}} - z\hat{\mathbf{k}}\right|^3} \\ &= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dz \rho \hat{\mathbf{k}} \times \hat{\mathbf{r}}}{\left|\rho\hat{\boldsymbol{\rho}} - z\hat{\mathbf{k}}\right|^3} \\ &= \frac{\mu_0 I \rho}{4\pi} \hat{\boldsymbol{\varphi}} \int_{-\infty}^{\infty} \frac{dz}{(\rho^2 + z^2)^{3/2}} \end{aligned}$$

Notice that the direction of the field circulates around the wire. Letting $z = \rho \tan\theta$ and changing variables,

$$dz = \frac{\rho}{\cos^2\theta} d\theta$$

and the integral gives

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dz}{(\rho^2 + z^2)^{3/2}} &= \int_{-\infty}^{\infty} \frac{\rho}{\cos^2\theta} d\theta \frac{1}{\left(\rho^2 + \rho^2 \tan^2\theta\right)^{3/2}} \\
&= \frac{1}{\rho^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta \\
&= \frac{1}{\rho^2} \sin\theta \Big|_{-\pi/2}^{\pi/2} \\
&= \frac{2}{\rho^2}
\end{aligned}$$

and therefore,

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0 I}{2\pi\rho} \hat{\boldsymbol{\varphi}}$$

3.3 Magnetic field of a current loop

Consider a single loop of thin wire of radius R lying in the xy plane, centered on the z-axis, and carrying a steady current I. Find the magnetic field at points along the z-axis.

There is no symmetry to the magnetic field strength, so we must use the Biot-Savart law. We may write the current densityas

$$\mathbf{J} = I\delta(\rho - R)\,\delta(z)\,\hat{\boldsymbol{\varphi}}$$

or simply use the resulting line charge form of the Biot-Savart law,

$$\mathbf{B}(\mathbf{x}_0) = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l} \times (\mathbf{x}_0 - \mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|^3}$$

The observation point $\mathbf{x}_0$ lies along the z-axis, so $\mathbf{x}_0 = z\hat{\mathbf{k}}$. The position of the current is $\mathbf{x} = R\hat{\boldsymbol{\rho}}$ as φ varies from 0 to π. This means that $d\mathbf{l} = Rd\varphi\hat{\boldsymbol{\varphi}}$. Assembling the parts,

$$\begin{aligned}
\mathbf{B}(\mathbf{x}_0) &= \frac{\mu_0}{4\pi} \int \frac{IRd\varphi\hat{\boldsymbol{\varphi}} \times \left(z\hat{\mathbf{k}} - \rho\hat{\boldsymbol{\rho}}\right)}{\left|z\hat{\mathbf{k}} - R\hat{\boldsymbol{\rho}}\right|^3} \\
&= \frac{\mu_0}{4\pi} \int \frac{IRd\varphi\left(z\hat{\boldsymbol{\rho}} + R\hat{\mathbf{k}}\right)}{(z^2 + R^2)^{3/2}} \\
&= \frac{\mu_0 IRz}{4\pi (z^2 + R^2)^{3/2}} \int \hat{\boldsymbol{\rho}} d\varphi + \frac{\mu_0 IR^2}{4\pi (z^2 + R^2)^{3/2}} \hat{\mathbf{k}} \int d\varphi
\end{aligned}$$

The first integral vanishes,

$$\begin{aligned}
\int_0^{2\pi} \hat{\boldsymbol{\rho}} d\varphi &= \int_0^{2\pi} \left(\hat{\mathbf{i}}\cos\varphi + \hat{\mathbf{j}}\sin\varphi\right) d\varphi \\
&= \left(\hat{\mathbf{i}}\sin\varphi - \hat{\mathbf{j}}\cos\varphi\right)\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

The second just gives 2π, so

$$\mathbf{B}(z) = \frac{\mu_0 I R^2}{2\left(z^2+R^2\right)^{3/2}}\hat{\mathbf{k}}$$