Hamilton-Jacobi theory

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1 Free particle

The simplest example is the case of a free particle, for which the Hamiltonian is

$$H = \frac{p^2}{2m}$$

and the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2$$

Let

$$S = f(x) - Et$$

Then $f(x)$ must satisfy

$$\frac{df}{dx} = \sqrt{2mE} = a$$

where E and a are constants. Therefore

$$f(x) = ax - c$$

where c is constant and we write the integration constant E in terms of the new (constant) momentum. Hamilton's principal function is therefore

$$S(x, q, t) = ax - \frac{a^2}{2m}t - c$$

We have no simple way to express this in terms of q, because the original coordinate x is cyclic. However, we know that the new Hamiltonian must vanish, so

$$\begin{aligned} K = 0 &= H + \frac{\partial S}{\partial t} \\ &= \frac{p^2}{2m} - \frac{a^2}{2m} \end{aligned}$$

so that $p = a$. This means that p is constant, and therefore equal to its initial value, making the initial momentum $\pi = a$. The principal function, dropping the irrelevant constant, is therefore

$$S(x, \pi, t) = \pi x - \frac{\pi^2}{2m}t$$

For a generating function of this type we set $f = -\pi q + S$ so that

$$\begin{aligned} pdx - Hdt &= \pi dq - Kdt + df \\ &= \pi dq - Kdt - \pi dq - qd\pi + \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial \pi}d\pi + \frac{\partial S}{\partial t}dt \end{aligned}$$

and we therefore have the relations

$$
\begin{aligned}
p &= \frac{\partial S}{\partial x} = \pi \\
q &= \frac{\partial S}{\partial \pi} = x - \frac{\pi}{m} t \\
K &= H + \frac{\partial S}{\partial t} = \frac{p^2}{2m} - \frac{\pi^2}{2m}
\end{aligned}
$$

Because $p = \pi$, the new Hamiltonian, K, is zero. This means that both q and π are constant. The solution for x and p follows immediately:

$$
\begin{aligned}
x &= q + \frac{\pi}{m} t \\
p &= \pi
\end{aligned}
$$

We see that the new canonical variables (q, π) are just the initial position and momentum of the motion, and therefore do determine the motion. The fact that knowing q and π is equivalent to knowing the full motion rests here on the fact that S generates motion along the classical path. In fact, given initial conditions (q, π), we can use Hamilton's principal function as a generating function but treat π as the *old* momentum and x as the *new* coordinate to reverse the process above and generate $x(t)$ and p.

2 Projectile motion

Consider a particle in a uniform gravitational field, with potential

$$V = mgz$$

The kinetic energy is

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

so taking the initial time to be t_0 =0, the action is given by

$$S = \int_0^t \left[\frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - mgz \right] dt$$

The conjugate momenta are then

$$
\begin{aligned}
p_x &= m\dot{x} \\
p_y &= m\dot{y} \\
p_z &= m\dot{z}
\end{aligned}
$$

and the Hamiltonian is

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz$$

Since x and y are cyclic, and $\frac{\partial H}{\partial t} = 0$, the corresponding momenta, p_x and p_y, are conserved, and the energy, $E = H$, is conserved.

The Hamilton-Jacobi equation is

$$\frac{1}{2m} \left(\left(\frac{\partial \mathcal{S}}{\partial x} \right)^2 + \left(\frac{\partial \mathcal{S}}{\partial z} \right)^2 + \left(\frac{\partial \mathcal{S}}{\partial z} \right)^2 \right) + mgz = -\frac{\partial \mathcal{S}}{\partial t}$$

This is completely separable. Writing

$$\mathcal{S} = \mathcal{S}_x(x) + \mathcal{S}_y(y) + \mathcal{S}_z(z) - Et$$

gives

$$\begin{aligned}
\frac{1}{2m}\left(\left(\frac{d\mathcal{S}_x}{dx}\right)^2 + \left(\frac{d\mathcal{S}_y}{dy}\right)^2 + \left(\frac{d\mathcal{S}_z}{dz}\right)^2\right) + mgz &= E \\
\left(\frac{d\mathcal{S}_x}{dx}\right)^2 + \left(\frac{d\mathcal{S}_y}{dy}\right)^2 + \left(\frac{d\mathcal{S}_z}{dz}\right)^2 + 2m^2gz &= 2mE
\end{aligned}$$

This is only possible if

$$\begin{aligned}
\left(\frac{d\mathcal{S}_x}{dx}\right)^2 &= \alpha^2 \\
\left(\frac{d\mathcal{S}_y}{dy}\right)^2 &= \beta^2
\end{aligned}$$

where α and β are constants, and

$$\left(\frac{d\mathcal{S}_z}{dz}\right)^2 + 2m^2gz = 2mE - \alpha^2 - \beta^2$$

The first two are immediately integrated to give

$$\begin{aligned}
\mathcal{S}_x &= \alpha x + c_1 \\
\mathcal{S}_y &= \beta y + c_2
\end{aligned}$$

Define $\gamma^2 = 2mE - \alpha^2 - \beta^2$, so that

$$\begin{aligned}
\frac{d\mathcal{S}_z}{dz} &= \sqrt{\gamma^2 - 2m^2gz} \\
\mathcal{S}_z &= \int_{z_0}^{z} \sqrt{\gamma^2 - 2m^2gz}\,dz
\end{aligned}$$

Substitute, $\zeta = \gamma^2 - 2m^2gz$, then

$$\begin{aligned}
\mathcal{S}_z &= -\frac{1}{2m^2g}\int_{z_0}^{z} \sqrt{\zeta}\,d\zeta \\
&= -\frac{1}{2m^2g}\frac{2}{3}\left.\zeta^{3/2}\right|_{z_0}^{z} \\
&= -\frac{1}{3m^2g}\left[\left(\gamma^2 - 2m^2gz\right)^{3/2} - \left(\gamma^2 - 2m^2gz_0\right)^{3/2}\right]
\end{aligned}$$

and Hamilton's principal function is therefore

$$\mathcal{S} = \alpha x + \beta y - \frac{1}{3m^2g}\left(\gamma^2 - 2m^2gz\right)^{3/2} - Et$$

where we drop the irrelevant constants.

Again using this as a generating function of type $\mathcal{S}(x_i, \pi_i, t)$, we have

$$\begin{aligned} p_i &= \frac{\partial S}{\partial x_i} \\ q_i &= \frac{\partial S}{\partial \pi_i} \\ K &= H + \frac{\partial S}{\partial t} \end{aligned}$$

The first equation gives

$$\begin{aligned} p_x &= \alpha \\ p_y &= \beta \\ p_z &= \sqrt{\gamma^2 - 2m^2 gz} \\ &= \sqrt{2m\left(E - \frac{p_x^2}{2m} - \frac{p_y^2}{2m} - mgz\right)} \end{aligned}$$

and the final shows that $H = E$, as expected. The energy may be written as

$$2mE = \alpha^2 + \beta^2 + \gamma^2$$

so that

$$p_z = \sqrt{\alpha^2 + \beta^2 + \gamma^2 - p_x^2 - p_y^2 - 2m^2 gz}$$

and

$$\mathcal{S} = \alpha x + \beta y - \frac{1}{3m^2 g}\left(\gamma^2 - 2m^2 gz\right)^{3/2} - \frac{1}{2m}\left(\alpha^2 + \beta^2 + \gamma^2\right) t$$

Taking the constants of integration (α, β, γ) as the new "momentum" variables, we have

$$\begin{aligned} q_x &= \frac{\partial S}{\partial \alpha} \\ &= x - \frac{\alpha}{m} t \\ q_y &= \frac{\partial S}{\partial \beta} \\ &= y - \frac{\beta}{m} t \\ q_z &= \frac{\partial S}{\partial \beta} \\ &= -\frac{\gamma}{m^2 g}\left(\gamma^2 - 2m^2 gz\right)^{1/2} - \frac{1}{m}\gamma t \end{aligned}$$

Finally, we invert these relations to find x, y, z as functions of the initial conditions and time:

$$\begin{aligned} x &= q_x + \frac{\alpha}{m} t \\ y &= q_y + \frac{\beta}{m} t \\ \left(q_z + \frac{1}{m}\gamma t\right)^2 &= \frac{\gamma^2}{m^4 g^2}\left(\gamma^2 - 2m^2 gz\right) \\ \left(q_z + \frac{1}{m}\gamma t\right)^2 &= \frac{\gamma^4}{m^4 g^2} - \frac{2\gamma^2}{m^2 g} z \end{aligned}$$

$$\begin{aligned}
\frac{2\gamma^2}{m^2 g} z &= \frac{\gamma^4}{m^4 g^2} - \left(q_z + \frac{1}{m}\gamma t\right)^2 \\
\frac{2\gamma^2}{m^2 g} z &= \frac{\gamma^4}{m^4 g^2} - \left(q_z^2 + \frac{2}{m} q_z \gamma t + \frac{1}{m^2}\gamma^2 t^2\right) \\
z &= \frac{m^2 g}{2\gamma^2}\left[\frac{\gamma^4}{m^4 g^2} - q_z^2 - \frac{2}{m} q_z \gamma t - \frac{1}{m^2}\gamma^2 t^2\right] \\
z &= \left(\frac{\gamma^2}{2m^2 g} - \frac{m^2 g q_z^2}{2\gamma^2}\right) - \frac{m g q_z}{\gamma} t - \frac{g}{2} t^2
\end{aligned}$$

and we may identify

$$\begin{aligned}
z_0 &= \frac{\gamma^2}{2m^2 g} - \frac{m^2 g q_z^2}{2\gamma^2} \\
\dot{z}_0 &= -\frac{m g q_z}{\gamma}
\end{aligned}$$

3 Simple harmonic oscillator

Consider a 1-dim simple harmonic oscillator, with action

$$S = \int \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2\right] dt$$

momentum,

$$p = m\dot{x}$$

and Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

The Hamiltonian-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \frac{1}{2} k x^2 = -\frac{\partial \mathcal{S}}{\partial t}$$

Write

$$\mathcal{S} = \mathcal{S}_x(x) - Et$$

to separate variables. This gives one integration constant, E, which is conveniently written as $E = \frac{\pi^2}{2m}$. Then the new variable π has units of momentum. Introducing $\omega = \sqrt{\frac{k}{m}}$ as well, the remaining part of the equation is then

$$\begin{aligned}
\left(\frac{d\mathcal{S}_x}{dx}\right)^2 + mkx^2 &= \pi^2 \\
\frac{d\mathcal{S}_x}{dx} &= \sqrt{\pi^2 - m^2\omega^2 x^2} \\
\mathcal{S}_x &= \int_{x_0}^{x} \sqrt{\pi^2 - m^2\omega^2 x^2}\, dx \\
&= \pi \int_{x_0}^{x} \sqrt{1 - \frac{m^2\omega^2 x^2}{\pi^2}}\, dx
\end{aligned}$$

and with $x = \frac{\pi}{m\omega}\sin\theta$ this becomes

$$\begin{aligned}
\mathcal{S}_x &= \pi \int\limits_{x_0}^{x} \sqrt{1-\sin^2\theta}\,\frac{\pi}{m\omega}\cos\theta d\theta \\
& \frac{\pi^2}{m\omega}\int\limits_{x_0}^{x}\cos^2\theta d\theta \\
&= \frac{\pi^2}{2m\omega}\int\limits_{x_0}^{x}\left(\cos 2\theta + 1\right) d\theta \\
&= \frac{\pi^2}{2m\omega}\left(\frac{1}{2}\sin 2\theta + \theta\right) \\
&= \frac{\pi^2}{2m\omega}\left(\sin\theta\cos\theta + \theta\right) \\
&= \frac{\pi^2}{2m\omega}\left(\frac{m\omega x}{\pi}\sqrt{1-\frac{m^2\omega^2 x^2}{\pi^2}} + \sin^{-1}\left(\frac{m\omega x}{\pi}\right)\right) \\
&= \frac{x}{2}\sqrt{\pi^2 - m^2\omega^2 x^2} + \frac{\pi^2}{2m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right)
\end{aligned}$$

Therefore,

$$\mathcal{S} = \frac{x}{2}\sqrt{\pi^2 - m^2\omega^2 x^2} + \frac{\pi^2}{2m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right) - \frac{\pi^2 t}{2m}$$

and this is a function of the the old position and the new momentum, $\mathcal{S}(x,\pi)$, so we haveTherefore,

$$\begin{aligned}
p &= \frac{\partial \mathcal{S}}{\partial x} \\
q &= -\frac{\partial \mathcal{S}}{\partial \pi} \\
K &= H + \frac{\partial \mathcal{S}}{\partial t}
\end{aligned}$$

We immediately have

$$\begin{aligned}
K &= H - \frac{\pi^2}{2m} \\
&= H - E \\
&= 0
\end{aligned}$$

so that Hamilton's equations give q and π constant. Then

$$\begin{aligned}
p &= \frac{\partial \mathcal{S}}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{2}\sqrt{\pi^2 - m^2\omega^2 x^2} + \frac{\pi^2}{2m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right) - \frac{\pi^2 t}{2m}\right) \\
&= \frac{1}{2}\sqrt{\pi^2 - m^2\omega^2 x^2} - \frac{2m^2\omega^2 x^2}{4\sqrt{\pi^2 - m^2\omega^2 x^2}} + \frac{\pi^2}{2m\omega}\frac{1}{\sqrt{1-\frac{m^2\omega^2 x^2}{\pi^2}}}\frac{m\omega}{\pi} \\
&= \frac{1}{2\sqrt{\pi^2 - m^2\omega^2 x^2}}\left(\pi^2 - m^2\omega^2 x^2 - m^2\omega^2 x^2 + \pi^2\right) \\
&= \sqrt{\pi^2 - m^2\omega^2 x^2}
\end{aligned}$$

which we recognize as the usual energy relationship

$$\begin{aligned} E &= \frac{\pi^2}{2m} \\ &= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \end{aligned}$$

Finally, to find the motion, we compute

$$\begin{aligned} q &= -\frac{\partial \mathcal{S}}{\partial \pi} \\ &= -\frac{\partial}{\partial \pi}\left(\frac{x}{2}\sqrt{\pi^2 - m^2\omega^2x^2} + \frac{\pi^2}{2m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right) - \frac{\pi^2 t}{2m}\right) \\ &= -\left(\frac{x\pi}{2\sqrt{\pi^2 - m^2\omega^2x^2}} + \frac{\pi}{m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right) + \frac{\pi^2}{2m\omega}\frac{1}{\sqrt{1 - \frac{m^2\omega^2x^2}{\pi^2}}}\left(-\frac{m\omega x}{\pi^2}\right) - \frac{\pi t}{m}\right) \\ &= -\frac{x\pi}{2\sqrt{\pi^2 - m^2\omega^2x^2}} - \frac{\pi}{m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right) + \frac{\pi x}{2\sqrt{\pi^2 - m^2\omega^2x^2}} + \frac{\pi t}{m} \\ q &= -\frac{\pi}{m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right) + \frac{\pi t}{m} \end{aligned}$$

Solving for x, we have

$$\begin{aligned} \frac{\pi}{m\omega}\sin^{-1}\left(\frac{m\omega x}{\pi}\right) &= -q + \frac{\pi t}{m} \\ \sin^{-1}\left(\frac{m\omega x}{\pi}\right) &= \frac{m\omega}{\pi}\left(-q + \frac{\pi t}{m}\right) \\ x &= \frac{\pi}{m\omega}\sin\left(\omega t - \frac{m\omega q}{\pi}\right) \end{aligned}$$

We may identify the amplitude and phase of the oscillator as

$$\begin{aligned} A &= \frac{\pi}{m\omega} \\ \varphi_0 &= \frac{m\omega q}{\pi} \end{aligned}$$

so that the position and momentum are

$$\begin{aligned} x(t) &= A\sin(\omega t - \varphi_0) \\ p(t) &= \sqrt{\pi^2 - m^2\omega^2x^2} \\ &= \sqrt{m^2\omega^2A^2 - m^2\omega^2A^2\sin^2(\omega t - \varphi_0)} \\ &= m\omega A\cos(\omega t - \varphi_0) \end{aligned}$$