Physics 3550, Fall 2012

Two Body, Central-Force Problem

Relevant Sections in Text: §8.1 – 8.7

Two Body, Central-Force Problem – Introduction.

I have already mentioned the two body central force problem several times. This is, of course, an important dynamical system since it represents in many ways the most fundamental kind of interaction between two bodies. For example, this interaction could be gravitational – relevant in astrophysics, or the interaction could be electromagnetic – relevant in atomic physics. There are other possibilities, too. For example, a simple model of strong interactions involves two-body central forces.

Here we shall begin a systematic study of this dynamical system. As we shall see, the conservation laws admitted by this system allow for a complete determination of the motion. Many of the topics we have been discussing in previous lectures come into play here. While this problem is *very* instructive and physically quite important, it is worth keeping in mind that the complete solvability of this system makes it an exceptional type of dynamical system. We cannot solve for the motion of a generic system as we do for the two body problem.

The two body problem involves a pair of particles with masses m_1 and m_2 described by a Lagrangian of the form:

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|).$$

Reflecting the fact that it describes a closed, Newtonian system, this Lagrangian is invariant under spatial translations, time translations, rotations, and boosts.* Thus we will have conservation of total energy, total momentum and total angular momentum for this system.

The EL equations are equivalent to (exercise)

$$m_1\ddot{\vec{r}}_1 = V'(|\vec{r}_1 - \vec{r}_2|)\,\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_1 - \vec{r}_2|},$$
$$m_2\ddot{\vec{r}}_2 = V'(|\vec{r}_1 - \vec{r}_2|)\,\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}.$$

* "Boosts" are transformations corresponding to changing to a reference frame moving with constant relative velocity. The totality of these symmetry transformations (spacetime translations, rotations, boosts) in non-relativistic Newtonian mechanics forms a group known as the *Galileo group.*

Here the prime on the potential energy function denotes a derivative with respect to its argument. The equations of motion consist of 6 coupled ODEs corresponding to the 6 degrees of freedom. In general these equations are non-linear. There is one exceptional case where the equations are linear. Can you see what this case is?

From these equations we see that the force exerted on each particle (i) is equal in magnitude and opposite in direction to that on the other particle (*i.e.,* this force obeys Newton's third law); (ii) is directed along a line joining the (instantaneous) particle positions, and (iii) has a magnitude depending only on the distance between the particles.

Of course, the standard example of such a two body problem arises in the motion of gravitating bodies ("Kepler problem"), or in the motion of a pair of electric charges ignoring magnetic and radiative effects. In each case the potential energy is of the form

$$V \propto \frac{1}{|\vec{r}_1 - \vec{r}_2|}.$$

The constant of proportionality depends upon the masses, charges, gravitational constant, *etc.*

Reduction to relative motion

Our first step toward solving the two body problem is to reduce the problem to that of the relative motion of the two bodies. The physical idea is simple. Given the homogeneity of space, the absolute position of either of the particles has no relevant meaning. Only the relative position should matter. This idea manifests itself as follows. Because the system is invariant under spatial translations, total momentum will be conserved. This means the motion of the center of mass will be at constant velocity in any inertial reference frame (IRF) (exercise). So, this conservation law tells us that 3 degrees of freedom move as a free particle. The remaining 3 degrees of freedom describe the relative motion of the particles; it is here where all the interesting dynamics lie. This effective reduction in degrees of freedom from 6 to 3 as arises from the conservation of total momentum (which is the center of mass momentum) of the system.

Let us now make the above comments mathematically precise. Define six generalized coordinates which are to replace the 6 Cartesian coordinates labeling the position of each of the bodies:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2},$$
$$\vec{r} = \vec{r}_1 - \vec{r}_2.$$

We call $\vec{R}$ the *center of mass position* and we call $\vec{r}$ the *relative position.* This is easily seen to be an invertible transformation. We have

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r}, \quad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r}.$$

Substituting these expressions (and their time derivatives) into the Lagrangian, we get

$$L = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}m\dot{\vec{r}}^2 - V(|\vec{r}|),$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

is called the *reduced mass* of the system.

We see that, in these coordinates, the Lagrangian is of the form

$$L(\vec{r}, \vec{R}, \dot{\vec{r}}, \dot{\vec{R}}) = L_{cm}(\dot{\vec{R}}) + L_{relative}(\vec{r}, \dot{\vec{r}}),$$

indicating that the center of mass motion $\vec{R}(t)$ and relative motion $\vec{r}(t)$ are completely decoupled (exercise). We can solve for the motion of $\vec{R}$ and $\vec{r}$ separately. The motion of $\vec{R}$ is obvious:

$$\vec{R}(t) = \vec{R}_0 + \vec{V}t.$$

By a suitable choice of inertial reference frame we can even set $\vec{R}(t) = 0$ without affecting $\vec{r}(t)$ (exercise). Put differently, up to a boost — which is a symmetry for this system — all solutions to the equations of motion can be obtained by setting $\vec{R}(t) = 0$ and solving for the relative motion. Once we have determined $\vec{r}(t)$, we can easily reconstruct $\vec{r}_1(t)$ and $\vec{r}_2(t)$ from $\vec{r}(t)$ and $\vec{R}(t)$ (exercise).

In any case, the only non-trivial problem to solve is the relative motion problem. We henceforth focus on the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(|\vec{r}|),$$

which describes the relative motion. Note that, mathematically speaking, this dynamical system is indistinguishable from a single particle moving in three dimensions subject to a central force field directed at fixed point in space. We have thus used the conservation of total momentum to reduce the analysis of the motion of two bodies interacting by a central force to the analysis of a system mathematically equivalent to that of a single particle moving in a central force field. All our subsequent analysis will apply to this physical system. Keep in mind that, in applications to the 2-body problem, the "particle" with mass m and position $\vec{r}$ is fictitious: its position is really the relative position of the two bodies, and the "particle" mass is really the reduced mass of the two body system.

Exercise: Show that when $m_2 >> m_1$ we can work in a reference frame in which m_2 is (approximately) at rest and we can view the reduced Lagrangian as describing the motion of m_1 in the fixed force field of m_2.

A small digression: Lagrangian Reduction

By a clever change of variables in the 2 body problem, $(\vec{r}_1, \vec{r}_2) \to (\vec{R}, \vec{r})$, we are able to obtain a Lagrangian in which three coordinates, namely (R_x, R_y, R_z), are cyclic. Thus the center of mass degrees of freedom "move" according to free particle EL equations and can be eliminated from the problem leaving us with a reduced problem involving 3 fewer degrees of freedom. You can see now why cyclic coordinates are also sometimes called "ignorable".

The reduction of the 2 body problem to a problem involving only 3 degrees of freedom by using conservation of total momentum is an instance of what is usually called *Lagrangian reduction*. This term describes the effective reduction in degrees of freedom afforded by a conservation law. A trivial example of Lagrangian reduction occurs already for motion in one-dimension q described by a time independent Lagrangian. The conservation of energy allows us to completely solve for $q(t)$ (up to quadrature)—thus leaving no more degrees of freedom to worry with. As we shall see, the complete integration of the EL equations for the 2-body problem is nothing but a sequence of Lagrangian reductions. There is a very general and powerful theory of reduction in mechanics (and field theory), but it will take us too far afield to develop it systematically. The best version of it takes place in the Hamiltonian formalism (to be discussed later), where the technique is called "symplectic reduction". Still, even though we will refrain from developing the general theory of Lagrangian reduction, because this is such an important tool in mechanics it is worth digressing for a moment to look at a couple of other examples of this phenomenon.

Example: Spherical Pendulum

The Lagrangian is

$$L = \frac{1}{2} m R^2 (\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) + mgR\cos\theta.$$

The coordinate ϕ is ignorable – it does not appear in L – and hence (exercise)

$$mR^2 \sin^2\theta \dot{\phi} = l = \text{constant}.$$

For any given l, this equation can be used to eliminate the ϕ degree of freedom in terms of an integral involving $\theta(t)$ (exercise). Thus we have a reduced problem involving a single degree of freedom θ. Once we find $\theta(t)$ we can easily obtain $\phi(t)$ by an integral, and the problem is solved.

The motion in θ, *i.e.,* $\theta(t)$, is determined by the equation of motion for θ with $\phi(t)$ eliminated using the conservation law above (exercise). This can be done because $\phi(t)$ does not appear in the equation, only $\dot{\phi}(t)$. Alternatively, we can eliminate $\dot{\phi}$ in the energy:

$$E = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} \frac{l^2}{mR^2} \sin^2\theta - mgR\cos\theta,$$

and use the conservation of energy to find $\theta(t)$ as described earlier. Note that it is precisely the symmetry (ϕ is cyclic) which guarantees that the conservation law can be used to completely eliminate ϕ from the θ equation of motion. Thus we use the conservation of angular momentum to reduce the two degrees of freedom of the pendulum to an effective one-dimensional system which can be solved via conservation of energy.

Example: Plane pendulum with moving point of support.

The Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\left(2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\right) - m_2 gl\cos\phi.$$

The coordinate x is ignorable – it does not appear L, so that the conservation law is

$$(m_1 + m_2)\dot{x} + (m_2 l\dot{\phi}\cos\phi) = \text{constant} \equiv p.$$

This conservation law can be used to eliminate $x(t)$ once p is specified and $\phi(t)$ is known. Thus we end up with a reduction to a single degree of freedom ϕ. As in our previous example, we can find the motion of ϕ by eliminating $\dot{x}$ from the energy in terms of $\phi(t)$ and the constant p and then using conservation of energy to find $\phi(t)$.

In the two body problem, we likewise used the conservation of total momentum – 3 conservation laws – to reduce the problem from 6 to 3 degrees of freedom. Unlike our two examples above, the particular geometry of center of mass and relative motion is such that this reduction is particularly simple: the two sets of degrees of freedom completely decouple. In general this decoupling need not occur, instead we get the scenario typified in the previous two examples. Indeed, we can use conservation laws to further reduce the relative motion problem, even though we don't get a decoupling.

Two Body Problem: Reduction to 1 radial degree of freedom.

We now return to our study of the 2 body problem, which has been reduced to the problem of a (fictitious) particle moving in a time independent central force field described by the Lagrangian

$$L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2}m\dot{\vec{r}}^2 - V(|\vec{r}|).$$

The EL equations are

$$m\ddot{\vec{r}} = -\nabla V = -V'(|\vec{r}|)\frac{\vec{r}}{|\vec{r}|}.$$

To get this far we have used the conservation of total momentum. We still have conservation of angular momentum and conservation of energy at our disposal. In this section we implement the reduction procedure associated with conservation of angular momentum.

Note that $L(\vec{r}, \dot{\vec{r}})$ given above is rotationally invariant about any axis through the center of force (exercise). Thus the angular momentum,

$$\vec{M} = \vec{r} \times \vec{p},$$

is conserved. Note that this the angular momentum of the *relative* motion. In terms of the positions of the two bodies we have

$$\vec{M} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 - \vec{r}_1 \times \vec{p}_2 - \vec{r}_2 \times \vec{p}_1.$$

The first two terms look like the total angular momentum of the system, but what about the other two terms? Recall that the angular momentum depends upon the choice of origin. A preferred origin has already shown itself: the center of mass frame. In the inertial reference frame in which the center of mass is at rest at the origin we have

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0.$$

In *this reference frame* – relative to the center of mass – it is easily seen that $\vec{M}$ is the total angular momentum of the two body system:

$$\vec{M} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2.$$

Since there are 3 conserved components in $\vec{M}$, it is tempting to believe we can use conservation of angular momentum alone to completely integrate the EL equations. This turns out not to be the case because of a technical subtlety: at any given point $\vec{r}$, only 2 components of angular momentum are linearly independent — this follows from the fact that

$$\vec{r} \cdot \vec{M} = 0.$$

For this reason we can use conservation of angular momentum to eliminate only 2 degrees of freedom. We could do this by writing out the equations and just computing things as we did in the Lagrangian reduction examples – those pendulum examples – we shall do this below. But, first, a more elegant and useful geometric point of view is worth exploring.

For any motion of the system, the vector $\vec{M}$ is fixed in space for all time. (Of course which vector this is depends upon the initial conditions, *i.e.,* upon which solution is being studied.) Because

$$\vec{r}(t) \cdot \vec{M} = 0 = \vec{p}(t) \cdot \vec{M},$$

it is easy to see that the motion $\vec{r}(t)$ of the "particle" is always in a plane orthogonal to $\vec{M}$, unless $\vec{M} = 0$. In the $\vec{M} = 0$ case there are three possibilities (1) $\vec{r} = 0$, which we don't allow; (2) $\vec{p} = 0$, which can't happen for more than a single instant of time; the acceleration of $\vec{r}$ is such that at the next instant of time $\vec{p} \neq 0$ and is parallel to $\vec{r}$; (3) $\vec{r}$ and $\vec{p}$ are proportional, in which case the particles are moving along the line joining them. This is, of course, also motion in a plane. So, let us suppose, without loss of generality, that we have chosen a plane for the orbit to lie in. This specification amounts to a choice of the allowed initial conditions. Alternatively, the initial conditions pick out a plane orthogonal to $\vec{M}$ that the motion will lie within. For $\vec{M} \neq 0$, this choice is equivalent to specifying the *direction* of the angular momentum vector. Although we are to imagine having made a definite choice for this plane, it is completely arbitrary and no special features of the choice will be used in what follows. Furthermore, the rotational symmetry which spawned the conservation law means any other choice of plane, *i.e.,* another set of initial conditions, can be obtained from the one we choose by a rotation. Since rotations map solutions to solutions,* we can get all solutions from the one obtained by restricting to a given plane by simply rotating the obtained solutions. So, we can restrict our study to motion in a plane with no essential loss of generality. Thus we have used conservation of the angular momentum *direction* to reduce the system by one degree of freedom, *i.e.,* we now have a problem of motion of a "particle" in a plane.

There is available a less elegant, but more straightforward, "brute force" approach to understanding this bit of reduction. Suppose we fix some initial conditions and want to see what the resulting motion looks like. Any given initial position and velocity will span a plane. Call this plane the x-y plane and construct spherical coordinates. The Lagrangian takes the form

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2) - V(r).$$

The EL equations are

$$m\ddot{r} - mr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + V'(r) = 0,$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2 \sin\theta\cos\theta\dot{\phi}^2 = 0,$$

$$\frac{d}{dt}(mr^2 \sin^2\theta\dot{\phi}) = 0.$$

From these equations it follows that all three components of $\vec{L}$ are conserved. The initial position and velocity lie in the x-y plane; this means that $L_x = L_y = 0$ initially and for all time. Expressed in spherical coordinates, we have

$$L_x = mr^2(\sin\phi\dot{\theta} - \frac{1}{2}\sin(2\theta)\cos\phi\dot{\phi}) = 0, \quad L_y = mr^2(\cos\phi\dot{\theta} - \frac{1}{2}\sin(2\theta)\sin\phi\dot{\phi}) = 0.$$

One solution of these conditions is $\dot{\theta} = \dot{\phi} = 0$, which is the $\vec{L} = 0$ case (straight line motion of the two particles). In this case we can assume our coordinates are chosen so that $\theta(t) = \frac{\pi}{2}$. Assuming $\vec{L} \neq 0$, we must have $\dot{\phi} \neq 0$ (exercise) and then the above

* Any symmetry of the Lagrangian transforms a solution of the EL equations into another solution of these same equations. It is a good exercise to prove this.

two conditions force $\theta(t) = \frac{\pi}{2}$. Thus we can always choose coordinates so that motion associated with any given initial conditions is in the x-y plane.

The equations of motion for $r(t)$ and $\phi(t)$, when restricted to the x-y plane, are (exercise)

$$m\ddot{r} = mr\dot{\phi}^2 - V'(r),$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0.$$

These equations can be derived from the Lagrangian (exercise)

$$L(r, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

This Lagrangian can be obtained by simply setting $\theta = \frac{\pi}{2}$ in the original Lagrangian, $L(\vec{r}, \dot{\vec{r}})$, after expressing the Lagrangian in spherical coordinates (exercise). That this is the correct procedure in this particular step is actually guaranteed by the general theory of Lagrangian reduction, but in general it is dangerous to simply eliminate degrees of freedom in the Lagrangian, since one can obtain incorrect EL equations by this procedure (see below). The reason it works in this case is that we are free to impose the holonomic constraint $\theta = \pi/2$ in the Lagrangian as usual when analyzing dynamical systems with constraints. Anyway, we have now reduced our analysis to that of a system with 2 degrees of freedom, namely (r, ϕ).

Let us summarize the situation thus far. We started with 6 degrees of freedom. The conservation of total momentum allowed us to reduce to 3 (relative) degrees of freedom. The conservation of the *direction* of angular momentum allows us to reduce to 2 degrees of freedom. The reduced system at this stage consists of a "particle" with the reduced mass m moving in a plane under the influence of a central force.

So far we have used the fact that the *direction* of $\vec{L}$ is a constant of motion, we still can use the fact that the magnitude of $\vec{L}$ is conserved. Having chosen our z-axis along $\vec{L}$, the magnitude of $\vec{L}$ is, of course, the z-component of angular momentum:

$$|\vec{L}| = mr^2\dot{\phi} = \text{constant} = L.$$

That this quantity *is* a constant of motion is easily seen directly from our latest Lagrangian, $L(r, \dot{r}, \dot{\phi})$, since ϕ is cyclic. As discussed in our previous examples, we can use this conservation law to reduce our analysis to that of a system with 1 degree of freedom. Specifically, we can use the conservation of L to eliminate $\phi(t)$ in terms of $r(t)$ and the constant L. We have

$$\dot{\phi} = \frac{L}{mr^2}, \quad \Longrightarrow \quad \phi(t) = \int_{t_0}^{t} ds\, \frac{L}{mr^2(s)}.$$

The reduced equation of motion for $r(t)$ is (exercise)

$$m\ddot{r} = -\frac{d}{dr}V_{\rm eff}(r),$$

where

$$V_{\rm eff} = V(r) + \frac{L^2}{2mr^2}.$$

This equation of motion can be derived from the Lagrangian for a system with a single degree of freedom (exercise):

$$L = \frac{1}{2}m\dot{r}^2 - V_{\rm eff}(r).$$

Note that this Lagrangian is **not** *obtained by using conservation of angular momentum to algebraically eliminate $\dot{\phi}$ from $L(r,\dot{r},\dot{\phi})$.*

We now have reduced the problem to an autonomous system with one degree of freedom. As we have discussed, we can handle this problem using conservation of energy. The total conserved energy of the system *is* correctly given by simple substitution for $\dot{\phi}$ in the conserved energy for the motion in the plane (exercise):

$$E = \frac{1}{2}m\dot{r}^2 + V_{\rm eff}.$$

The contribution of the angular momentum to the effective potential energy represents an effective *repulsive* force (exercise) due to the angular motion of the "particle". For this reason, the term $\frac{L^2}{2mr^2}$ is often called the "centrifugal energy".

We have reduced the dynamics of the two body problem to that of a system of one degree of freedom, r, described by the Lagrangian

$$L = \frac{1}{2}m\dot{r}^2 - V_{\rm eff}(r),$$

where

$$V_{\rm eff} = V(r) + \frac{L^2}{2mr^2}.$$

Using the conservation of energy, in which

$$E = \frac{1}{2}m\dot{r}^2 + V_{\rm eff},$$

we can express the motion of the radial degree of freedom via (exercise)

$$t - t_0 = \pm\int_{r_0}^{r} \frac{1}{\sqrt{\frac{2}{m}(E - V(x)) - \frac{L^2}{m^2x^2}}}\,dx.$$

Thus using conservation of total momentum, angular momentum, and energy we are able to solve the equations of motion "up to quadrature".

The orbits

The preceding reduction procedure provides a method for analyzing the motion in time of the two bodies. There is no closed form expression, however, for this motion – it is simply too complex to be expressed using elementary analytic functions. It is possible, however, to find the (relative) motion at the expense of dropping all information about how the path is traced out in time.

To begin, we have the relation between

$$\dot{\phi} = \frac{L}{mr^2(t)},$$

which means

$$d\phi = \frac{L}{mr^2(t)} dt.$$

Provided $L \neq 0$, we can parametrize the curves in configuration space with ϕ instead of t. Going back to the conservation of energy formula and trading dt for $d\phi$ we have (exercise)

$$d\phi = \pm \frac{L}{r^2\sqrt{2m(E - V(r)) - \frac{L^2}{r^2}}} dr. \tag{1}$$

Thus we can express the angular motion as the integral:

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{L}{x^2\sqrt{2m(E - V(x)) - \frac{L^2}{x^2}}} dx. \tag{2}$$

Here we have absorbed the $\pm$ into the choice of L. Typically, it is far easier to perform this integral than those needed to get the motion in time. Mathematically, this is because of the extra $\frac{1}{x^2}$ factor in the integrand. Physically this is because, by dropping the dependence on time, the integral does not have to convey so much information about the motion of the "particle".

We can understand this integral expression in another way. Consider the radial equation of motion:

$$\ddot{r} = -V'_{\text{eff}}(r).$$

Make the change of variables $d\phi = \frac{L}{mr^2(t)} dt$. After a little algebra we get the differential equation for the orbit $r = r(\phi)$:

$$\frac{L}{r^2}\frac{d}{d\phi}\left(\frac{L}{mr^2}\frac{dr}{d\phi}\right) = -V'_{\text{eff}}(r).$$

The solution $r = r(\phi)$ is, in general, determined by the integral expression (2) given above. This follows from the first integral for the orbit equation

$$\frac{1}{2}\frac{L^2}{m^2 r^4} r'^2 + V_{\text{eff}} = E,$$

which is just conservation of energy expressed in terms of the orbit $r(\phi)$.

A useful way to write the orbit equations is to make the change of variables,

$$u = \frac{1}{r},$$

After a bit of algebra, we then get

$$u'' = -u + \frac{m}{l^2 u^2} V'.$$

The relation between r and ϕ from the last integral gives the path traced out by the "particle" via $\phi = \phi(r)$. If desired, we can invert to get $r = r(\phi)$, which is the *orbit* of the particle. Knowledge of the orbit (path) carries no information about how the path is traced out in time. This is obtained from the integral relating r and t, giving $r = r(t)$ and $\phi = \phi(t) = \phi(r(t))$. Note that our elimination of the time in favor of the radius is possible only when $\dot{r} \neq 0$. Consequently we cannot use this trick to get the orbit when the particle's radius is at an equilibrium point of the effective potential. Of course this is either a turning point, which is easily handled by continuity of the orbit (exercise), or one has $r =$ constant, *i.e.,* circular motion. In this case it is hardly necessary to use the integral representation of the path in order to understand the orbit!

For a given choice of the energy E, the values of r where

$$E = V_{\text{eff}}(r),$$

define the turning points of the radial motion. At the turning points the radial velocity vanishes; it changes sign as the particle passes through the turning point. Of course, the "particle" doesn't actually come to rest at the turning point since its angular motion is monotonic in time, as can be seen from the conservation of angular momentum formula (exercise). If E is such that there are two turning points then we have a bound state and the motion is confined to an annular region in the plane of motion; we have orbital motion. If there is only one turning point the motion is unbound and we have a scattering situation, which we may discuss later.

For now, let us restrict our attention to bound motion. If the potential energy function is such that V_{eff} has a minimum, and if E takes that minimum value, then the radial motion is that of stable equilibrium. In this case $r =$ constant and the motion is circular and, of course, periodic. More generally, however, the motion is considerably more complicated. Even though the bound motion is periodic in the radial variable, this does not mean that the motion in the plane is periodic. This is because the time it takes for ϕ to change by 2π need not be the same as the time it takes for r to pass through one cycle. In detail, during the time that the radial variable passes through one cycle, the angle changes by (exercise)

$$\Delta\phi = 2\int_{r_0}^{r_1} \frac{L}{x^2\sqrt{2m(E - U(x)) - \frac{L^2}{x^2}}}\, dx,$$

where r_0 and r_1 are the turning points of the radial motion. For the path of the "particle" in the plane to define a closed orbit $\Delta\phi$ must be a rational multiple of 2π:

$$\Delta\phi = \frac{p}{q}(2\pi),$$

so that after q cycles in the radial motion the angular variable will have made a net change of $2\pi p$, *i.e.,* the "particle" will have made p revolutions.

Generically, that is, for "almost all" potentials $V(r)$, the condition for closure of the orbit will *not* be satisfied. Thus, for a typical central force problem the relative motion will correspond to that of a "particle" that follows a curve about the origin which is bounded by r_0 and r_1 and eventually passes through every point in this annular region. Exceptions to this occur when $V \propto \frac{1}{r}$ (Kepler problem; to be discussed shortly) or $V \propto r^2$ (isotropic oscillator). We shall see shortly that there is a hidden symmetry and conservation law in these exceptional cases that "explains" the closure of the orbits.

Finally we note that thanks to the repulsive effect of the "centrifugal energy", the "particle" cannot reach the center of force $r = 0$ (where the 2 bodies collide) unless the central force is sufficiently attractive — and singular — as $r \to 0$. To see this, simply consider the formula for the energy. Assuming that r is changing, and using the fact that the radial contribution to the kinetic energy is positive, we have the inequality (exercise)

$$r^2V(r) + \frac{L^2}{2m} < Er^2$$

or

$$r^2(V(r) - E) < -\frac{L^2}{2m}.$$

This means that as r approaches zero $V \sim -cr^{-\alpha}$, up to an additive constant, with $c > 0$ appropriately chosen, and $\alpha \geq 2$. Of course, if $L = 0$ this condition has much weaker implications for the potential.

Exercise: What is the nature of the motion if $L = 0$?

Kepler Problem

The Kepler problem refers to the case where the potential energy is of the form

$$V(r) = \pm\frac{\alpha}{r}.$$

Here $\alpha > 0$, the plus sign corresponds to a repulsive force, such as between two charges of the same sign. The minus sign corresponds to an attractive force, such as the electric force between two oppositely charged particles, or the gravitational force between two point masses. In either case, it is straightforward to find the path the "particle"* takes, while the motion in time is more difficult to display explicitly.

We will consider the case of an attractive force. I will leave it as an exercise for you to translate, where appropriate, to the results for the case of a repulsive force. Note that the potential used in the Kepler problem is not sufficiently attractive to allow the "particle" to hit the origin when $L \neq 0$ (see the discussion of this point in the previous lecture). Thus, except in the exceptional case $L = 0$, the two bodies never collide. We will assume $L \neq 0$ in what follows.

Exercise: Describe the motion of the two bodies in the $L = 0$ case.

Let us first consider some general properties of the motion. The effective potential energy for the radial motion,

$$V_{\text{eff}}(r) = -\frac{\alpha}{r} + \frac{L^2}{2mr^2},$$

tends to $+\infty$ as $r \to 0$, and tends to zero from negative values as $r \to \infty$, thus both bound (orbital, 2 turning points) and unbound (scattering, 1 turning point) motions are possible. There is a single, global minimum to the effective potential energy (exercise):

$$(V_{\text{eff}})_{\text{min}} = -\frac{m\alpha^2}{2L^2}.$$

This represents a stable equilibrium for the *radial* motion. If the energy and angular momentum are related via

$$E = (V_{\text{eff}})_{\text{min}}$$

we get circular motion at a radius which you can work out from the previous equation to be

$$r = \frac{L^2}{m\alpha} = -\frac{\alpha}{2E}.$$

* Keep in mind we are still considering the motion of two bodies; $\mathbf{r}$ is the relative position of the particles and r is the distance between them.

Let us next get an explicit formula for the orbital path. The orbital path is obtained by performing the integral relating ϕ and r; we get

$$\phi = \cos^{-1}\left\{\frac{\frac{L}{r} - \frac{m\alpha}{L}}{\sqrt{2mE + \frac{m^2\alpha^2}{L^2}}}\right\} + \text{constant}.$$

With a little algebra you can see that this relationship between r and ϕ is of the form

$$\frac{1}{r} = A\cos(\phi + \delta) + \frac{\alpha m}{L^2}.$$

There is another way to see this result. Recall we made the change of variables

$$d\phi = \frac{L}{mr^2(t)}dt, \quad u = \frac{1}{r}$$

to get the differential equation for the orbit $u = u(\phi)$. For the Kepler problem this takes the form

$$u'' = -u + \frac{\alpha m}{L^2}.$$

Evidently the function $w = u - \frac{\alpha m}{L^2}$ satisfies

$$w'' = -w,$$

so that

$$w(\phi) = A\cos(\phi + \delta),$$

hence

$$u(\phi) = A\cos(\phi + \delta) + \frac{\alpha m}{L^2},$$

and finally

$$\frac{1}{r} = A\cos(\phi + \delta) + \frac{\alpha m}{L^2}.$$

To keep things simple, let us adjust the origin of ϕ so that the integration constant vanishes. Define

$$p = \frac{L^2}{m\alpha}, \quad e = \sqrt{1 + \frac{2L^2E}{m\alpha^2}}.$$

Then we can write the path as (exercise)

$$\frac{p}{r} = 1 + e\cos\phi.$$

This is the equation of a *conic section* with the origin (where $r = 0$) being one of the foci. The parameter e is the *eccentricity* of the conic section. Evidently, our choice of origin for

ϕ corresponds to the minimum value of r. If the motion is that of a body (*e.g.,* the Earth) and the sun, the point where $\phi = 0$ is the *perihelion*.

We see that the path of motion for the (attractive) Kepler problem is either circular ($e = 0$), elliptic ($e < 1$), parabolic ($e = 1$), hyperbolic ($e > 1$). From the expression for the eccentricity in terms of the energy, we see that circular motion does indeed correspond to the minimum of the potential energy (exercise):

$$E = -\frac{m\alpha^2}{2L^2},$$

as expected. Elliptic motion corresponds to

$$-\frac{m\alpha^2}{2L^2} < E < 0.$$

Parabolic motion corresponds to $E = 0$. And hyperbolic motion corresponds to $E > 0$.

If one body is very much more massive than the other, then it is easy to see that this massive body will remain (approximately) at the $r = 0$ focus of the conic section. For bound motion of a planet around the sun we thus recover Kepler's law of orbits.

For elliptical motion ($E < 0$) the semi-major axis has length (exercise)

$$a = \frac{p}{1 - e^2} = \frac{\alpha}{2|E|},$$

which only depends upon the energy of the particle. The semi-minor axis has length

$$b = \frac{p}{\sqrt{1 - e^2}} = \frac{L}{\sqrt{2m|E|}},$$

which depends upon the energy and angular momentum.

The period of the motion for elliptical (or circular) orbits can be instructively computed as follows. Recall the equation defining the conserved magnitude of the angular momentum:

$$L = mr^2\dot{\phi}.$$

For infinitesimal displacements in time dt, it is straightforward to see that the quantity

$$df = \frac{1}{2}r^2 d\phi$$

is the area contained between the position vectors at time t and time $t + dt$. We call

$$\dot{f} = \frac{1}{2}r^2\dot{\phi}$$

the *areal velocity* since it represents the rate at which area is swept out by the position vector in time. Conservation of angular momentum tells us that the areal velocity is constant. Thus we recover Kepler's law of areas: equal areas are swept out in equal times. Note that this result follows from conservation of angular momentum and so is not specific to the Kepler problem; it holds for any 2-body central force problem. Now, we have

$$L = 2m\dot{f}.$$

In one period the position vector sweeps out the area πab of the ellipse. Thus the period T is given by (exercise)

$$T = \frac{2m\pi ab}{L} = \pi\alpha\sqrt{\frac{m}{2|E|^3}}.$$

Note that the period only depends upon the energy. Since the semi-major axis is proportional to $\frac{1}{|E|}$, we see Kepler's law of periods being verified (exercise).

When $E \geq 0$ the motion is unbounded. When $E > 0$, we have $e > 0$ and the motion is hyperbolic with perihelion

$$r_{\text{min}} = \frac{\alpha(e-1)}{2E},$$

which depends on both energy and angular momentum (exercise). When $E = 0$, we have $e = 1$ and

$$r_{\text{min}} = \frac{L^2}{2m\alpha}.$$

Note that parabolic motion has the "particle" having zero velocity asymptotically (exercise).

We have been focusing primarily on the path of the "particle". Its motion in time is governed by Kepler's law of areas, as we have discussed. Unfortunately, a completely explicit analytical description of the motion is hard to come by. In particular, the integral giving $t(r)$ is tractable enough, but the problem of inversion to get $r(t)$ (which is also needed to get $\phi(t)$) is analytically somewhat involved.