Physics 3550, Fall 2012

Motion near equilibrium - Small Oscillations

Relevant Sections in Text: §5.1 – 5.6, §11.1 – 11.3

Motion near equilibrium − 1 degree of freedom

One of the most important situations in physics is motion near equilibrium. Motion near stable equilibrium can always be decomposed into the motion of harmonic oscillators. From a pendulum in a clock, to a bridge swaying in the breeze, to photons in quantum electrodynamics, the harmonic approximation for motion near equilibrium comes into play. We will now touch on some of the key ideas.

You will recall that a Newtonian particle has an *equilibrium point* $\vec{r}_0$ at locations where the force on the particle vanishes $\vec{F}(\vec{r}_0) = 0$. In terms of a potential energy functions $V(\vec{r})$, an equilibrium point is a critical point of the potential energy:

$$\vec{\nabla} V(\vec{r}_0) = 0.$$

So, let us begin by considering the motion of a one-dimensional system near a critical point of the potential energy. While only the simplest systems can be reduced to one degree of freedom, this case is still particularly important because – as we shall soon see – multi-dimensional systems (in the vicinity of stable equilibrium) can be reduced to multiple copies of the one dimensional case.

We assume the Lagrangian is of the form:

$$L = \frac{1}{2}a(q)\dot{q}^2 - V(q).$$

Note we assume (for now) that the system is closed, *i.e.*, energy is conserved. This assumption is built into the Lagrangian — there is no explicit time dependence. Let q_0 be a critical point of the potential, so that $V'(q_0) = 0$. Let us approximate the motion by assuming that $x := q - q_0$ is "small". We expand the Lagrangian in a Taylor series in x, keeping only the first non-trivial terms. Using

$$a(q) \approx a(q0) + \ldots, \quad \dot{q} = \dot{x}, \quad V(q) \approx V(q_0) + V'(q_0)x + \frac{1}{2}V''(q_0)x^2 + \ldots$$

we get

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \ldots.$$

Here we have set $m = a(q_0)$ and $k = V''(q_0)$. Of course, the kinetic energy must be positive, so that $m > 0$. Note that we also have dropped the irrelevant constant $V(q_0)$. If desired we can adjust the reference point of potential energy so that this constant is zero.

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The EL equations are (in the domain of validity of the approximation)

$$m\ddot{x} = -kx.$$

This equation is easily integrated:

$$x = Re(Ae^{i\omega t}), \quad \text{when } k \neq 0.$$

$$x = at + b, \quad \text{if } k = 0,$$

where $\omega = \pm\sqrt{\frac{k}{m}}$ and A is a complex constant encoding the two real integration constants, which can be fixed by initial conditions.

Taking the real part of the exponential expresses the solution in terms of a cosine. An alternative form of the exponential solution corresponds to using a sine:

$$x = Im(Ae^{i\omega t}), \quad \text{when } k \neq 0.$$

If $k > 0$, then q_0 is a point of stable equilibrium, and we get harmonic motion. In particular, if x is small initially and the initial velocity is sufficiently small, then $x(t)$ remains small (exercise), so that our approximation is self-consistent. On the other hand, if $k \leq 0$, then the motion of the particle need not maintain our approximation of small x; our approximation is not self-consistent and must be abandoned after a very short time. Of course, the case $k < 0$ corresponds to unstable equilibrium, for which a small perturbation leads to a rapid motion away from equilibrium. If $k = 0$, then the critical point q_0 is not a maximum or minimum but is a saddle point ("neutral equilibrium"); our approximation again becomes invalid, although the time scale for this is larger than the case of unstable equilibrium.

Thus in a neighborhood of a point of stable equilibrium it is consistent to make the *harmonic approximation* to the potential and kinetic energies. In the harmonic approximation the motion of the system is mathematically the same as that of a simple harmonic oscillator. If the initial displacement and velocity are small enough, the motion will be well-approximated by harmonic motion near equilibrium.

A very familiar example of all of this is the planar pendulum of mass m and length l for which $q = \theta$ is the deflection from a vertical position. We have

$$V(\theta) = mgl(1 - \cos\theta),$$

and

$$a(\theta) = \frac{1}{2}ml^2.$$

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The Lagrangian in the harmonic approximation near equilibrium at $\theta = 0$ is (exercise)

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{mgl}{2}\theta^2,$$

so that the harmonic motion has "mass" ml^2 and angular frequency" $\omega = \sqrt{\frac{g}{l}}$. (Exercise: what happens to motion near $\theta = \pi$?)

Example

Consider a mass m which is constrained to move on a straight line. The mass is bound to a fixed point by harmonic force with potential energy $V = \frac{1}{2}K(r-R)^2$, where K is a constant, r is the distance of the particle to the fixed point. The distance from the point to the line is $l > R$. A mechanical model of this system is a mass sliding on a straight track; the mass being connected to a fixed point by a spring. Our goal is to find the stable equilibrium position(s) and compute the frequency of small oscillations about the equilibrium. Evidently, $r^2 = x^2 + l^2$, where x is the position of the particle along the given line, with $x = 0$ the location at distance l from the center of force. Using x as the generalized coordinate, the Lagrangian for this system is (exercise)

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}K(\sqrt{x^2 + l^2} - R)^2.$$

You can check that $x = 0$ is the only point of stable equilibrium. (As an exercise you can check that the point $x = 0$ is a point of unstable equilibrium if we assume $R \geq l$.) You can expand the potential to second order to find that (exercise)

$$k = \frac{K(l-R)}{l},$$

so that

$$\omega = \sqrt{\frac{K(l-R)}{lm}}.$$

Note that $K(l-R)$ is the force needed to move a particle from $r = R$ to the point $r = l$ (exercise) in the presence of the potential V.

Example

We return to our example of a plane pendulum of length l with horizontally moving point of support. The mass m_1 (the point of support) has position x, and the angular deflection of the pendulum (mass m_2) is denoted by ϕ. The kinetic energy is

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi),$$

and the potential energy is

$$V = -m_2 g l \cos \phi.$$

We have already seen that we can use conservation of the momentum conjugate to x to effect a Lagrangian reduction which eliminates the x degree of freedom. In detail, the conservation law we need is

$$P = (m_1 + m_2)\dot{x} + m_2 l \dot{\phi} \cos \phi = \text{constant}.$$

This conservation law is just the conservation of the x component of the center of mass of the system (exercise).

Let us compute the frequency of small oscillations of the pendulum in the reference frame in which $P = 0$, so that (exercise)

$$(m_1 + m_2)x + m_2 l \sin \phi = \text{constant}.$$

In this reference frame, which is the rest frame of the x component of the center of mass, the reduced kinetic energy is (exercise)

$$T = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \frac{m_2}{m_1 + m_2} \cos^2 \phi \right),$$

and the reduced potential energy is (exercise)

$$V = -m_2 g l \cos \phi.$$

Clearly $\phi = 0$ is an equilibrium point (exercise). Expanding in powers of ϕ we get, in the harmonic approximation,

$$T \approx \frac{1}{2} \frac{m_1 m_2 l^2}{m_1 + m_2} \dot{\phi}^2,$$

$$V \approx \frac{1}{2} m_2 g l \phi^2.$$

It is now straightforward to see that the frequency of small oscillations is (exercise)

$$\omega = \sqrt{\frac{g(m_1 + m_2)}{m_1 l}}.$$

Of course, in a generic inertial reference frame the motion of the system in the harmonic approximation is small oscillations at the above frequency superimposed with a uniform translation of the center of mass along the x direction.

As a sanity check, let us consider the limit in which $\frac{m_2}{m_1} << 1$, *i.e.,* the mass at the point of support is becoming large. Physically, we expect that the point of support moves

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with a uniform translation along x. In the rest frame of the point of support (which is now the approximate center of mass) we have a traditional plane pendulum problem. In this limit we get (exercise)

$$\omega \approx \sqrt{\frac{g}{l}},$$

as expected.

Damped Oscillations

Usually, motion near equilibrium of a realistic system includes dissipation of energy into an environment. This is why oscillating systems in the real world have a decreasing amplitude in time and eventually come to rest. The mathematical description of this damping can become rather intricate depending upon the causes of the energy dissipation. A simple and important case is the damping caused by motion in a fluid at sufficiently small speeds. We will consider this case, where the dissipation is proportional to velocity. There are other possibilities, of course.

The equation of motion is

$$m\ddot{x} + b\dot{x} + kx = 0,$$

with $b > 0$ controlling the dissipation. You can see that the effect of dissipation is always a deceleration. A simple example where such dissipation occurs is motion of an oscillating mass in a fluid. Another interesting example is in an LRC electrical circuit.* If $q(t)$ is the charge on the capacitor at time t, we have

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0.$$

You can see the resistance is the source of the energy dissipation. This is the usual I^2R power dissipation.

Let us return to the general form of the damped oscillator equation. If we divide the equation of motion for $x(t)$ by m, we have

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0,$$

where $\beta = \frac{b}{2m}$. This is a linear ODE with constant coefficients. A standard approach to solving such an equation is to try to find 2 independent solutions of the form

$$x(t) = De^{rt},$$

where D and r are constants. Substituting this into the differential equation is straightforward and gives an equation for r:

$$r^2 + 2\beta r + \omega^2 = 0,$$

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* L is inductance. C is capacitance. R is resistance.

which is easy to solve using the quadratic formula:

$$r = -\beta \pm \sqrt{\beta^2 - \omega^2}.$$

Our two independent solutions are now

$$x(t) = D_1 e^{-\beta t + \sqrt{\beta^2 - \omega^2}\, t} + D_2 e^{-\beta t - \sqrt{\beta^2 - \omega^2}\, t}.$$

The constants D_1 and D_2 are determined by – and determine – initial position and velocity. If $\beta = 0$, we see harmonic motion at frequency ω. If $|\beta| < |\omega|$, we get damped, oscillatory motion. This case is called "under-damped". If $|\beta| \geq |\omega|$ we get purely damped motion. This case is called "over-damped". You will explore this all a bit more in the homework.

Forced Oscillations

A typical scenario in which small oscillations are relevant is where one has a system in stable equilibrium which is subjected to an external force $\vec{F}$ which moves the system from equilibrium. (A very important real-world example: a child being pushed on a swing.) We allow this force to be time dependent. This introduces an additional potential energy term V_1 to the quadratically approximated Lagrangian given by

$$V_1 = -\vec{F}(t) \cdot \vec{x}.$$

Sticking with one-dimensional motion, we thus consider the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + xF(t),$$

where $k > 0$. The equations of motion are that of a harmonic oscillator subjected to an external, time varying force $F(t)$ (exercise):

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}.$$

Of course, we must assume that F remains suitably small and varies in time so that the solutions do not violate the approximation needed for their validity.

This inhomogeneous differential equation can be directly integrated as follows. Define

$$\xi(t) = \dot{x}(t) + i\omega x(t).$$

In terms of this complex variable the equation of motion takes the form (exercise)

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}.$$

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You can easily see that, when $F = 0$, the solution is of the form $Ae^{i\omega t}$, where A is a constant. So try a solution of the form

$$\xi(t) = A(t)e^{i\omega t}.$$

Plugging into the ODE, we find that A satisfies

$$\dot{A} = e^{-i\omega t}\frac{F(t)}{m},$$

which has solution (exercise)

$$A(t) = \int dt \, \frac{1}{m}F(t)e^{-i\omega t} + \text{ constant}.$$

Putting this all together, we see that the general solution of the forced oscillator equation is (exercise)

$$x(t) = \frac{1}{\omega}Im(\xi(t))$$
$$= \frac{1}{\omega}Im\left[e^{i\omega t}(B + \int \frac{1}{m}F(t)e^{-i\omega t}\, dt)\right],$$

where B is an arbitrary complex constant.

Notice that the solution for $x(t)$ is the sum of two terms:

$$x(t) = x_H(t) + x_I(t),$$

where

$$x_H(t) = Im(\frac{B}{\omega}e^{i\omega t}),$$

and

$$x_I(t) = \frac{1}{\omega}Im\left[e^{i\omega t}\int \frac{1}{m}F(t)e^{-i\omega t}\, dt\right]$$

You will recognize $x_H(t)$ as the general solution to the un-forced harmonic oscillator equation. You can easily check that $x_I(t)$ is a solution to the forced harmonic oscillator equation. What we have here is a very general kind of result, valid when the equation being solved is *linear, inhomogeneous*. The general solution of a linear, inhomogeneous equation is the sum of *any* particular solution and the general solution of the associated homogeneous equation.

As an example of all this, suppose that

$$F(t) = f\cos(\gamma t),$$

where f is a constant. Then, provided $\gamma^2 \neq \omega^2$, the general solution is of the form (exercise)

$$x(t) = a\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t),$$

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where a and α are real constants. These constants are fixed by initial conditions, as usual.

We see that in this case the motion is a superposition of two oscillations at the two frequencies ω and γ inherent in the problem. The relative importance of the forced oscillation component depends, of course, on the the size of f, but also on the relative magnitudes of ω and γ.

When $\omega \to \gamma$ the forced oscillation amplitude diverges and our form of the solution given above becomes invalid; this situation is called *resonance*. To get the correct solution in this case we set $\gamma = \omega$ in our integral expression or the general solution. We then get a solution of the form (exercise)

$$x(t) = a\cos(\omega t + \alpha) + \frac{f}{2m\omega}t\sin\omega t,$$

where, again, a and α are constants. Note the linear growth in t, which eventually destroys the harmonic approximation.

A simple example of resonance is when you push someone on a swing. If you time your pushes just right you can increase the amplitude of the swing.

Some detailed analysis

We can get a useful picture of the behavior of the system near resonance by an approximation scheme. Let

$$\gamma = \omega + \epsilon,$$

where $\epsilon << \omega$. Write the general solution for $x(t)$ (off resonance) in the complex form (exercise):

$$\xi(t) \approx (A + Be^{i\epsilon t})e^{i\omega t}.$$

Over one period, $\frac{2\pi}{\omega}$, the amplitude $C = |A + Be^{i\epsilon t}|$ changes very little. Thus the motion is approximately that of free oscillation with a slowly varying amplitude. In particular, the amplitude is of the form (exercise)

$$C = \sqrt{a^2 + b^2 + 2ab\cos(\epsilon t + \phi)},$$

where $A = ae^{i\alpha}$, $B = be^{i\beta}$, and $\phi = \beta - \alpha$. Thus the amplitude varies (slowly) between the values $|a + b|$ and $|a - b|$. The oscillatory behavior is said to exhibit "beats".

Typically, a general force $F(t)$ can be Fourier analyzed into sinusoidal components. Likewise, we can Fourier analyze the solution $x(t)$. We can view the above example as illustrating the behavior of a typical Fourier component. The general motion of the system is then a superposition of motions such as given above (exercise).

Finally, let us note that since the Lagrangian for a system executing forced oscillations is explicitly time dependent (provided $\frac{dF}{dt} \neq 0$), there will be no conservation of energy for the oscillator. This should not surprise you, since the oscillator is clearly exchanging energy with its environment. We can compute the energy transferred during a time interval (t_1, t_2) by noting that the oscillator energy can be written as

$$E = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) = \frac{1}{2}m|\xi|^2,$$

and then using our explicit formula for $\xi(t)$,

$$\xi = e^{i\omega t}\left(B + \int_0^t \frac{1}{m}F(t)e^{-i\omega t}\,dt\right)$$

to compute the energy at time t. For example, let us suppose that the system is at equilibrium before $t = 0$, a force acts for a period of time after $t = 0$ after which the force is zero again. Then $B = 0$ and the change in the oscillator energy can be written as (exercise)

$$\Delta E = \frac{1}{2m}\left|\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt\right|^2.$$

Thus, the energy transfer is controlled by the absolute value of the Fourier component of the force with frequency ω. If the time during which the force acts is small compared to $\frac{1}{\omega}$, then $e^{i\omega t}$ is approximately constant in the integral, and hence

$$\Delta E \approx \frac{1}{2m}\left|\int_{-\infty}^{\infty} F(t)\,dt\right|^2.$$

Here the change in energy is controlled solely by the impulse imparted by the force since the time scale is so short that no appreciable change in potential energy occurs while the force acts. In the limit where

$$F(t) = f\delta(t - t_0),$$

this approximation becomes exact (exercise).

Finally, we remark that one can treat the forced oscillator with damping included in a similar way to what was done above. See your text for details.

More than one degree of freedom

Here we consider a couple of important and illustrative examples of oscillations with more than one degree of freedom. In general if

$$V = V(q^1, q^2, \ldots, q^n)$$

is the potential energy function for a Newtonian system with n degrees of freedom. Then equilibrium q_0 is defined by

$$\frac{\partial V}{\partial q^i}(q_0) = 0, \quad i = 1, 2, \ldots, n.$$

and the small oscillation approximation involves the Taylor series of the potential energy to second order. Ignoring the zeroth order (constant) term, the first order term vanishes, and we have near equilibrium

$$V \approx \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial q^i \partial q^j}(q_0) x^i x^j, \quad x^i = q^i - q_0^i.$$

So, in this approximation the potential energy becomes a second order polynomial in the displacement from equilibrium. We will briefly explore a couple of standard situations which arise for this approximation.

Isotropic and anisotropic oscillators

We consider a system with two degrees of freedom (x, y) whose Lagrangian near equilibrium takes the form

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k_x x^2 - \frac{1}{2} k_y y^2, \quad k_x, k_y \text{ constants}$$

This corresponds to a matrix of second derivatives of the form

$$\frac{\partial^2 V}{\partial q^i \partial q^j}(q_0) = \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix}.$$

This system is known as a two-dimensional* *anisotropic oscillator*. An example where such a Lagrangian would be used is when studying a marble rolling near the bottom of a bowl which is sharply curved in one direction and less sharply curved in an orthogonal direction. Another example of such a Lagrangian (when generalized to 3-d) arises in condensed matter physics when studying the motion of an atom when displaced from equilibrium in a crystal which is not isotropic.

The equations of motion for this Lagrangian are easily seen to be just two, decoupled harmonic oscillator equations:

$$m\ddot{x} + k_x x = 0, \quad m\ddot{y} + k_y y = 0.$$

The solutions are

$$x(t) = A \cos(\omega_x t + \alpha), \quad y(t) = B \cos(\omega_y t + \beta),$$

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* I think you can easily generalize everything I will do here to more than two degrees of freedom.

where

$$\omega_x = \sqrt{\frac{k_x}{m}}, \quad \omega_y = \sqrt{\frac{k_y}{m}}$$

and A, B, α, β are constants which can be determined by initial conditions. The motion in each direction is simple (harmonic), but the combined motion can be quite complex. To be sure, with initial position and velocity in one of the directions set to zero, the motion is just harmonic oscillations in the other direction. But, in the generic case, where ω_x and ω_y are two random constants and the initial conditions are not fine tuned in any way, the particle wanders in the plane with a motion which never repeats itself! It can be shown that if ω_x and ω_y are *rationally related* – ω_x/ω_y is a rational number – then the motion is periodic. If $\omega_x = \omega_y$ then we speak of an *isotropic oscillator*. Now the motion is much more regular. Indeed you can check that, with $\omega_x = \omega_y$ the motion is generically an ellipse. See your text for details. The isotropic oscillator provides an example of a rotationally invariant central force; we shall revisit the isotropic oscillator when we study the 2-body central force problem.

Coupled oscillators

A very important result is illustrated in the next example: Motion near (stable) equilibrium can always be expressed as n decoupled oscilations, as in the last section, where n is the number of degrees of freedom of the system. The variables that effect this decoupling represent the *normal modes* of vibration. The associated frequencies are called *characteristic frequencies*.

Let us consider a Lagrangian of the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega_0^2(x^2 + y^2) - \frac{1}{2}m\alpha^2(x - y)^2$$
$$= \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 - (\omega_0^2 + \alpha^2)(x^2 + y^2) + 2\alpha^2 xy\right]$$

This system can be viewed as two identical one-dimensional harmonic oscillators (natural frequency ω_0) with a coupling by a harmonic force (natural frequency α) (exercise). So, a toy model for this system could be two pendulums connected by a spring, or two masses each connected by identical springs to two walls and inter-connected by a different spring. In any case, this is the Lagrangian for motion near equilibrium whenever the potential energy is such that:

$$\frac{\partial^2 V}{\partial q^i \partial q^j}(q_0) = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \begin{pmatrix} m(\omega_0^2 + \alpha^2) & -m\alpha^2 \\ -m\alpha^2 & m(\omega_0^2 + \alpha^2) \end{pmatrix}.$$

The equations of motion can be written as:

$$\ddot{x} + (\omega_0^2 + \alpha^2)x - \alpha^2 y = 0, \quad \ddot{y} + (\omega_0^2 + \alpha^2)y - \alpha^2 x = 0.$$

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As you can see, the "spring" characterized by the frequency α couples the two one-dimensional oscillators. How to solve these coupled, linear, homogeneous, second-order, constant coefficent, ordinary differential equations? A simple, linear change of coordinates will do the trick. Set

$$q_1 = x + y, \quad q_2 = x - y.$$

Add and subtract the two equations of motion to get

$$\ddot{q}_1 + \omega_0^2 q_1 = 0,$$

and

$$\ddot{q}_2 + (\omega_0^2 + 2\alpha^2)q_2 = 0.$$

The change of variables has decoupled the two ODEs and, moreover, has yield two 1-d harmonic oscillator equations!

So, in terms of the variables q_1 and q_2 the motion is mathematically that of an anisotropic two-dimensional oscillator – two decoupled simple harmonic oscillations as we treated in the previous subsection. We have the general solution

$$q_1(t) = C_1 \cos(\omega_0 t + \beta_1), \quad q_2(t) = C_2 \cos(\sqrt{\omega_0^2 + 2\alpha^2}\, t + \beta_2),$$

or

$$x(t) = \frac{1}{\sqrt{2}}\Big(C_1 \cos(\omega_1 t + \beta_1) + C_2 \cos(\omega_2 t + \beta_2)\Big)$$

$$y(t) = \frac{1}{\sqrt{2}}\Big(C_1 \cos(\omega_1 t + \beta_1) - C_2 \cos(\omega_2 t + \beta_2)\Big).$$

It can be shown that this type of result – the motion can be viewed as a combination of independent harmonic oscillations — arises whenever one approximates motion near equilibrium as we have done here, for any number of degrees of freedom! The harmonic motions are called *normal modes* of vibration and the associated frequencies are the *characteristic frequencies*.

Evidently, the normal mode at frequency ω_0 corresponds to the two masses moving exactly in phase, so the coupling does not come into play, while the oscillation at frequency $\sqrt{\omega_0^2 + 2\alpha^2}$ has the two masses moving exactly out of phase. The general motion of the system is a superposition of these two normal modes; the particular superposition which occurs is determined by initial conditions. In particular, if we choose the initial conditions just right we can have motion which is purely one or the other normal mode. For example, normal mode q_2 is excited by itself if one picks initial x and y to be equal and opposite and with vanishing initial velocity. Can you see this? For generic initial conditions the motion can be quite complicated as we discussed for the anisotropic oscillator.