Physics 3550, Fall 2012

Energy.

Relevant Sections in Text: Chapter 4

Energy: Inspiring Motivational Remarks

I probably don't need to impress upon you the importance of the concept of energy in analyzing/understanding physical systems. Energy can be used to understand a wide range of phenomenon, from atomic spectra, to the motion of the planets, to current trends in the Earth's climate.

Once again, the importance of the concept of energy stems from its conservation. Many interactions can be largely, if not completely, understood by analyzing the flow/exchange of energy. For many complex systems the only useful analysis one can do is based upon energy considerations. Let us begin by considering the most fundamental form of energy, *kinetic energy*, which characterizes the energy associated to motion.

Kinetic Energy.

As you well know, the kinetic energy of a particle is proportional to the squared-speed,

$$T = \frac{1}{2}mv^2.$$

For a system of N particles, the kinetic energy of the system is obtained by adding up the individual kinetic energies:

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2.$$

Let us focus on a single particle for simplicity. If the particle is doing anything interesting, its velocity will be changing. Hence, the kinetic energy is probably changing, too. The time rate of change of kinetic energy is easy to characterize. We have

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = m\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{F}\cdot\vec{v}.$$

The quantity on the right is the *power* delivered to the system (the particle).

Let us write this relationship as

$$dT = \vec{F}\cdot d\vec{r},$$

and consider the integral over the path taken by the particle from an initial point $\vec{r}_1$ to a final point $\vec{r}_2$. We get

$$\Delta T = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}.$$

The right hand side of this formula is *the work done on the system by the force* in moving from $\vec{r}_1$ to $\vec{r}_2$:

$$W \equiv \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}.$$

The result we just obtained, the change in kinetic energy is the work done, is the famous and very important *work-energy theorem.* The work-energy theorem is Newton's second law in terms of energy. If the work is positive, the kinetic energy of the system (particle) increases – work is done *by* the environment *on* the system. If the work is negative, the kinetic energy of system (particle) decreases – work is done *on* the environment *by* the system.

Note we can also write this theorem as

$$\Delta T = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = W.$$

In many applications this is the easiest way to compute the work. Let us illustrate the work-energy theorem with a couple of examples.

Consider a ball thrown horizontally off of a cliff with initial speed v_0. After a time T it hits the ground. Let us compute the work and the change in kinetic energy. First, let's compute the work. The simplest way is to use the time intergral of the power. Orienting our x axis horizontally and the y axis vertically, we have

$$\vec{v} = v_0\hat{\imath} - gt\hat{\jmath}, \quad \vec{F} = -mg\hat{\jmath}.$$

$$\vec{F} \cdot \vec{v} = mg^2 t.$$

$$W = \int_0^T dt\, mg^2 t = \frac{1}{2} mg^2 T^2.$$

Let us compute the change in kinetic energy:

$$\Delta T = T_{final} - T_{initial} = \frac{1}{2} m(v_0^2 + g^2 T^2) - \frac{1}{2} m v_0^2 = \frac{1}{2} mg^2 T^2 = W.$$

It is interesting and instructive to note that this is the same work and change in kinetic energy you would get if you dropped the ball straight down for the same amount of time.

Here is another example. Consider an electric charge moving under the influence of a magnetic field. Let us compute the work. Again, it is simplest to start from the power. We have

$$\vec{F} \cdot \vec{v} = q(\vec{v} \times \vec{B}) \cdot \vec{v}.$$

Now we recall that vector $\vec{a} \times \vec{b}$ is always perpendicular to each of $\vec{a}$ and $\vec{b}$. Thus

$$\vec{F} \cdot \vec{v} = 0,$$

and the work is zero for any path the particle takes!. This means that the *magnetic field does no work*. Motion of a charged particle in a magnetic field must be with a constant speed!*

A final example. A particle slides down a plane inclined at an angle α relative to the horizontal. If the initial height of the particle is h, what is the work done? We will use the "force times displacement" form of the work for this one. Let us characterize the displacement. We have

$$d\vec{r} = dx\hat{i} + dy\hat{j} = dr\cos\alpha\hat{i} - dr\sin\alpha\hat{j}$$

As for the force, we have

$$\vec{F} = \vec{F}_{normal} - mg\hat{j}.$$

Here we have indicated the normal force which keeps the particle on the plane. Note that $\vec{F}_{normal}\cdot d\vec{r} = 0$, so it does no work. We have

$$\vec{F}\cdot d\vec{r} = mg\sin\alpha dr = mgdy.$$

Thus

$$W = \int_0^{\frac{h}{\sin\alpha}} mg\sin\alpha\, dr = \int_0^h mg\, dy = mgh.$$

Conservative forces, potential energy, total energy

Not all forces are created equal. For some forces the work done in moving between two points does not depend upon what path the system took to get there! Let us see this with an example. First, consider a force for which the work done *does* depend upon the path, not just the initial and final points. Suppose we slide a body on the ground from a point $x = 0$ to a point $x = 1$ under the influence of friction $\vec{f}$. We do this in two ways. First, slide the body at constant speed in a straight line from $x = 0$ to $x = L$. Second, we slide the body at constant speed from $x = 0$ to $x = 2L$ then back to $x = L$. We compute the work done by friction for these two scenarios. The key fact about friction we will use here is that f is a constant, and $\vec{f}\cdot dr = -fdr$. For the first scenario we have

$$W_1 = -\int_0^L f dx = -fL.$$

For the second scenario we have

$$W_2 = -\int_0^{2L} f dx + \int_{2L}^L f dx = -3fL.$$

* Here is a paradox: You can certainly move things around with a magnet. How can that be if the magnetic field does no work?

So, even though the particle starts and ends in the same places, the work is different for the two paths, $W_1 \neq W_2$.

Now let us consider a similar situation, but instead of using friction, we will use the gravitational force $\vec{F} = -mg\hat{i}$ near the surface of the Earth. We throw a ball in the air and compute the work done by gravity as the ball gets to a height L. We suppose that the initial speed is such that the ball passes the height L, goes up a bit more, then falls back to height L and to the ground. So the ball gets to height L by two different paths. Let us compute the work done for each path. First, we work out the case where L is obtained on the way up:

$$W_1 = \int_0^L (-mg)dx = -mgL.$$

Now we consider the case where the particle goes to a height h, then falls back to L:

$$W_2 = \int_0^h (-mg)dx - \int_h^L (mg)dx = -mgh - mg(L-h) = -mgL.$$

We see that $W_1 = W_2$ in this case.

Now for a definition. When a force is such that: (1) the force only depends upon position, and (2) the work done is independent of the path taken between two given points, we say the force is *conservative*. Gravity is a conservative force (I will prove this shortly). Friction is not a conservative force it depends upon velocity and it fails condition (2) (just proved above).

Conservative forces lead to *conservation of energy* because one can keep track of the work done just by knowing the position of the particle. Again, we begin by considering a single particle for simplicity. Let the net force on the particle be conservative. Fix once and for all a *reference point* $\vec{r}_0$ in space. Because the force is conservative, the work done in going between the reference point and any other point $\vec{r}$ only depends upon the choice of $\vec{r}$. Given the particle's position $\vec{r}$ we define the *potential energy* $U(\vec{r})$ as the work done by the force in moving the particle from $\vec{r}$ to $\vec{r}_0$. Because the force is conservative, we can compute this work using any path we like and we always get the same result. We have

$$U(\vec{r}) = \int_{\vec{r}}^{\vec{r}_0} \vec{F} \cdot d\vec{r} = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}.$$

The last equality shows that the potential energy at $\vec{r}$ is minus the work done *on* the system in going from the reference point to $\vec{r}$. Alternatively, the potential energy is the work done *by* the system as it moves from the reference point to the point of interest. As the particle moves, its position and velocity change in time, of course. But the work-energy theorem tells us that, if the particle moves from position $\vec{r}_1$ to $\vec{r}_2$, then its change in kinetic energy is the work done, and the work done can be expressed in terms of potential energy. This suggests that a combination of kinetic and potential energies will be unchanged as the particle moves. This combination is called the (total) *energy*, and is defined as

$$E(\vec{r},\vec{v}) = T + U = \frac{1}{2}mv^2 + U(\vec{r}).$$

To see that the energy is conserved, consider its change, ΔE as the particle moves from position $\vec{r}_1$ to $\vec{r}_2$. The kinetic energy changes by ΔT. Since the work done does not depend upon the path. We can imagine the particle first goes from $\vec{r}_1$ to $\vec{r}_0$, then it goes from $\vec{r}_0$ to $\vec{r}_1$. The work in each case can be expressed in terms of the potential energy

$$\Delta T = T_2 - T_1 = W = U(\vec{r}_1) - U(\vec{r}_2) = -\Delta U.$$

So, we have

$$\Delta E = \Delta T + \Delta U = 0.$$

You have, of course, used this conservation law many times in introductory physics, so I won't belabor the simple examples right now.

Physicists generally believe that all forces are, ultimately, conservative. Non-conservative forces like friction arise only when one models complex conservative forces in a simple way thereby ignoring some of the energy accounting.

The potential energy is a function whose value at a point $\vec{r}$ is always defined by a line integral from some fixed reference point $\vec{r}_0$ to $\vec{r}$. The reference point is the location where the potential energy vanishes. The conservative nature of the force guarantees that the integral is independent of the path chosen. The result *does* depend upon the choice of reference point, however. Suppose we use a different choice of reference point, $\vec{r}_1$, say. Let U_1 be the potential energy associated with this reference point. We can compare the potential energies as follows. Consider the values of $U(\vec{r})$ and $U_1(\vec{r})$. We define $U(\vec{r})$ by a line integral along some path C from $\vec{r}_0$ to $\vec{r}$. Let us pick the path defining $U_1(\vec{r})$ to be a path C_1 which joins $\vec{r}_1$ to $\vec{r}_0$, then follows the same path C used for U to get to $\vec{r}$. We then get

$$U_1(\vec{r}) = -\int_{\vec{r}_1}^{\vec{r}_0} \vec{F}\cdot d\vec{r} - \int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r} = U(\vec{r}) + (const.),$$

where the constant is minus the work done by the force going from $\vec{r}_0$ to $\vec{r}_1$. The point here is that the freedom to choose a reference point is the same as the freedom to add a constant to the potential energy function. Clearly, adding a constant to the definition of U does not alter the conservation of total energy. Note, however, that the freedom to add a constant to U means that the value of U (and hence the value of E) has no direct physical significance. Only the *difference* in energies has a physical meaning.*

* This is a Newtonian mechanics result only. Einstein's theory of gravity – contained in his general theory of relativity – gives an absolute significance to energy since the presence of energy causes a curvature of spacetime.

Let us also note that when we restrict attention to one-dimensional motion, all one-dimensional time independent and velocity independent forces are conservative. This can be seen as follows. Let us call the one dimension x. The force is $\vec{F} = F(x)\hat{\imath}$. The work done in getting from $x = a$ to $x = b$ is an integral of the form

$$W = \int_a^b dx\, F(x).$$

As you know, if $x = c$ is any other point we have the basic calculus result

$$\int_a^b dx\, F(x) = \int_a^c dx\, F(x) + \int_c^b dx\, F(x).$$

This result implies that no matter how the particle gets from a to b the work is the same. Thus the work done is independent of the path and the force is conservative.†

Potential energy: some examples

Here let me show you some important examples of potential energy.

Free particle

Here the force is zero – clearly a conservative force! The infinitesimal work vanishes, $\vec{F} \cdot d\vec{r} = 0$, so the (definite) integral defining the work does to. The potential energy is zero.

Uniform force field

Here the magnitude F and direction of $\vec{F}$ are constant. This is easily seen to be a conservative force. Call the reference point the origin. Pick the z axis along the force direction, $\vec{F} = F\hat{k}$. In this coordinate system we have $\vec{F} \cdot d\vec{r} = F dz$ and the work integral is mathematically the same as that of a one-dimensional system, which is always conservative as we have seen. (A simpler proof can be obtained using the results of the next section.) For the potential energy, picking a straight line path from the reference point to the point of interest, we have

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} = \vec{F} \cdot (\vec{r}_0 - \vec{r}) = -\vec{F} \cdot \vec{r} + (const.).$$

In our adapted coordinate system, with the z axis along $\vec{F}$, we have, up to an additive constant, $U = -Fz$, which is a familiar kind of formula, *e.g.,* let $F_z = -mg$.

† Question for you: In light of this result, how is it that friction manages to be non-conservative?

One dimensional Harmonic Oscillator

Recall that the one dimensional harmonic oscillator has a restoring force of the form $F_x = -kx$, where the origin $x = 0$ is the equilibrium position. This force is conservative, as discussed above. Choosing the equilibrium position as the reference point, and a straight path to the point of interest we have

$$U(x) = -\int_0^x dx(-kx) = \frac{1}{2}kx^2.$$

Newtonian Gravity - one particle

Recall that the gravitational force on a particle of mass m located at $\vec{r}$ due a particle of mass M located at $\vec{R}$ is given by

$$\vec{F} = G\frac{mM}{d^3}(\vec{R} - \vec{r}), \quad d = |\vec{r} - \vec{R}|.$$

For starters, let us suppose that $M >> m$ so that we can work in a reference frame where M does not move. Place the origin of our coordinates at this position. Then we have

$$\vec{F} = -G\frac{mM}{r^3}\vec{r}.$$

We shall easily see that this force is conservative below. Let us assume it for now; we get to pick any path we like to compute the work and the potential energy function. To compute a potential energy function at $\vec{r}$, we fix our reference point and we integrate first along an angular path along a sphere centered at the origin of our coordinate system until we are on the line joining $\vec{r}$ and the origin. We then integrate radially to $\vec{r}$. Since the force is in the radial direction only, the work done when moving on the sphere vanishes. Evidently, the potential energy only depends upon the radius r and is given by

$$U(\vec{r}) = \int_{r_0}^{r} dr'\, G\frac{mM}{r'^2} = -GmM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

Note that we have taken account of the fact that the force vector field points in the negative radial direction. Traditionally, the reference point for this force is place "at infinity", in which case we have the familiar formula.

$$U(\vec{r}) = -\frac{GmM}{r}.$$

A closer look at conservative force fields: the gradient and the curl

Mathematically, a conservative force is a vector field on Euclidean space with the special property that its line integral between two points does not depend upon the path used to connect the two points. There is a simpler (but fully equivalent) characterization of a conservative force field which dispenses with all the line integrals. The idea is this. Pick two points $\vec{r}_1$ and $\vec{r}_2$. Compute the work done in getting from $\vec{r}_1$ to $\vec{r}_2$ using two different paths, say, $\vec{r}_a(t)$ and $\vec{r}_b(t)$. These two curves start at the same place, and they end at the same place. They define a closed loop C. The difference between the line integrals – the difference in the work done along the two paths – is given by the line integral of the force around the closed loop. If the force is conservative this closed loop integral must vanish. Now I invoke another fundamental result from vector analysis, known as *Stokes Theorem.** If C is an oriented closed curve enclosing an oriented area A with unit normal $\hat{n}$, then for any vector field $\vec{F}$:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_A (\nabla \times \vec{F}) \cdot \hat{n}\, dA.$$

Here we have used the *curl* of a vector field:

$$\nabla \times \vec{F} = (\partial_y F_z - \partial_z F_y)\hat{\imath} + (\partial_z F_x - \partial_x F_z)\hat{\jmath} + (\partial_x F_y - \partial_y F_x)\hat{k}.$$

This result allows us to prove that the force is conservative (the integral on the left side vanishes for any C) if and only if the curl of $\vec{F}$ vanishes:

$$\nabla \times \vec{F} = 0 \quad \Longleftrightarrow \quad \text{conservative force.}$$

It is now trivial to see that any uniform force field (*e.g.,* the usual model for gravity near the surface of the Earth) is conservative since all its components are constants. It is also trivial to see why one dimensional systems with time and velocity independent forces are conservative (exercise!).

With a bit of algebra the more realistic gravitational force law,

$$\vec{F} = \frac{k}{r^3}\vec{r},$$

can be checked to be conservative. Let me illustrate the computation by computing the x component of the curl. We have

$$\begin{aligned}(\nabla \times \vec{F})_x &= \partial_y F_z - \partial_z F_y \\ &= k\partial_y\Big(\frac{z}{(x^2+y^2+z^2)^{3/2}}\Big) - k\partial_z\Big(\frac{y}{(x^2+y^2+z^2)^{3/2}}\Big).\end{aligned}$$

You can easily check that this vanishes.

Here is a good exercise for you to try: Show that the force field

$$\vec{F} = y\hat{\imath} - x\hat{\jmath}$$

* Stokes theorem is at the heart of Faraday's law in electrodynamics.

is *not* conservative. This vector field is the prototype *non-conservative* vector field. Note that it's lines of force are circles; you can see why we call $\nabla \times \vec{F}$ the "curl".

We have seen that if the work done by a particular force is independent of the path then we can define a potential energy function and a conserved energy. We have also seen that if the work done by a particular force is independent of the path, then the curl of the force vector field vanishes. We can close the logical loop here by presenting one more very important result from vector analysis: The curl of a vector field $\vec{V}$ vanishes, if and only if the vector field can be written* as the *gradient* of a function ψ:

$$\nabla \times \vec{V} = 0 \quad \Longleftrightarrow \quad \vec{V} = \nabla\psi \equiv \partial_x\psi\,\hat{i} + \partial_y\psi\,\hat{j} + \partial_z\psi\,\hat{k}.$$

The proof that a gradient of a function defines a conservative force follows from the important vector identity

$$\nabla \times (\nabla\psi) = 0.$$

The converse claim takes some more work; I will not do it here. The point of this result for us is that a conservative force must be the gradient of a function and that function is (minus) the potential energy:

$$\vec{F} = -\nabla U, \quad \text{conservative force.}$$

A formula of this type should not surprise you since U is defined by integrating $\vec{F}$ – it makes sense that $\vec{F}$ should be defined by differentiating U. Let us just see how the formula works by checking one component of the curl of the gradient:

$$[\nabla \times (\nabla\psi)]_x = \partial_y(\nabla\psi)_z - \partial_z(\nabla\psi)_y = \partial_y\partial_z\psi - \partial_z\partial_y\psi = 0.$$

Let's check a few of our examples above.

Uniform force field

$$U(\vec{r}) = -\vec{F}\cdot\vec{r} = -F_x x - F_y y - F_z z, \quad F_x, F_y, F_z \ \textit{constants}$$

$$-\nabla U = -\left\{\partial_x(-\vec{F}\cdot\vec{r})\hat{i} + \partial_y(-\vec{F}\cdot\vec{r})\hat{j} + \partial_z((-\vec{F}\cdot\vec{r})\hat{k}\right\} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}.$$

* This result holds with appropriate boundary conditions. For simplicity, we will not go into this here.

One-dimensional oscillator

$$U(\vec{r}) = \frac{1}{2}kx^2,$$
$$-\nabla u = -(\partial_x U)\,\hat{i} = -kx\,\hat{i}.$$

Newtonian gravity – one particle

$$U(\vec{r}) = -\frac{GmM}{r},$$
$$-\nabla U = GmM\left\{(\partial_x \frac{1}{r})\,\hat{i} + (\partial_y \frac{1}{r})\,\hat{j} + (\partial_z \frac{1}{r})\,\hat{k}\right\} = -G\frac{mM}{r^3}\vec{r}.$$

Here we have used

$$\partial_x \frac{1}{r} = \partial_x \frac{1}{(x^2+y^2+z^2)^{1/2}} = -\frac{x}{(x^2+y^2+z^2)^{3/2}},$$

with similar results for the y and z derivatives.

Conservation of energy and one-dimensional systems

We now turn our attention to a class of dynamical systems which can always be understood in great detail: the *conservative system with one degree of freedom*. For conservative systems with one degree of freedom the motion is determined by conservation of energy.

Often times the study of motion of a dynamical system can be reduced to that of a one-dimensional system. For example, the motion of a plane pendulum reduces to dynamics on a circle. Another example that we shall look at in detail is the 2-body central force problem. Finding the motion of this system amounts to solving for the motion of the relative separation of the two bodies, which is ultimately a one-dimensional problem. No matter how the one-dimensional system arises, in very general circumstances we can completely solve the equations of motion.

Let us consider systems whose energy takes the form

$$E = \frac{1}{2}a(q)\dot{q}^2 + V(q),$$

where q is some coordinate for the system and a, V are some given functions. Of course, when $a(q) = m$ we have the usual kinetic energy for one-dimensional motion. A less trivial example arises when we consider a particle moving on a curve $y = 0$, $z = f(x)$ under the influence of gravity (a roller coaster!). This system has (exercise)

$$a(q,t) = m(1 + f'^2)$$

and

$$V(q) = mgf(q),$$

where we have chosen the position on the x-axis to be q. Here we see how the kinetic energy can become a function of the coordinate.

Our main result is the following: For systems with energy of the indicated form, the motion is completely determined by conservation of energy. Indeed, we have

$$\frac{1}{2}a(q(t))\left(\frac{dq}{dt}\right)^2 + V(q(t)) = \text{constant} \equiv E.$$

For a given value of E, we thus get a first-order (albeit non-linear) ODE for $q(t)$. Since the equations of motion *a la* Newton are second-order we have, in effect, used conservation of energy to integrate the equations of motion once. For this reason one often calls E a *first integral* of the equations of motion.

We can solve this first-order ODE as follows. Fix a numerical value for E; this is equivalent to partially fixing the initial conditions.* We then rearrange the energy conservation equation given above into the form

$$dt = \pm\sqrt{\frac{a(q)}{2(E - V(q))}}\, dq,$$

so that, upon integrating both sides,

$$t - t_0 = \pm\int_{q_0}^{q}\sqrt{\frac{a(x)}{2(E - V(x))}}\, dx.$$

Here q_0 is the other integration constant (besides E) which arises when solving a second order ODE. t_0 is a free parameter corresponding to a choice of initial time; when $t = t_0$ we have $q = q_0$. Note also that we have two possible signs which can occur when taking the square root. This amounts to whether the motion increases q with an increase in time, or decreases q with an increase in time. Both behaviors are possible in general; the boundary between these two types of motion is a time where the velocity vanishes. By specifying the $\pm$ sign, the value of q_0, and the value of E we have completely specified the solution. These choices are equivalent to specifying the initial values of q and dq/dt (exercise).

By performing the indicated integral we get a formula $t = t(q)$, which also depends upon the initial position, initial time, and energy. To get the solution of the equations of motion, $q = q(t)$, we need to invert this formula. This can be done provided $\frac{dt}{dq}$ is finite

* The equation of motion is a single second order ODE. Solutions are uniquely determined by 2 numbers, corresponding, *e.g.,* to the initial values of position and velocity. By fixing a value for the energy we are putting one condition on the two initial values.

and non-vanishing (exercise). This is guaranteed by the equations of motion away from points where $E = V$, *i.e.,* where the kinetic energy vanishes. As we shall see, these are isolated points. Away from such points we can solve (in principle) for $q = q(t)$, which also depends upon the initial position, initial time, and energy. By continuity we get the solution everywhere. We thus get the solution to the equations of motion depending upon two constants of integration (q_0 and E). We also have the $\pm$ sign in the solution. Specifying q_0 and E and a choice of sign is equivalent to specifying the initial position and velocity (exercise). Thus we obtain the general solution to the equations of motion.

Motion in One Dimension: Examples

The integral formula we obtained from conservation of energy specifies the solution—q versus t—for any one-dimensional (conservative) system. Generically, the integral formula is the best you can do. For sufficiently simple forms of $a(q)$ and $V(q)$ a closed form expression for the motion of the system can be found, *i.e.,* the integration can be carried out explicitly and one can rearrange things to get $q = q(t)$. Let us look at some elementary examples.

Example: Free Particle

Here $V(q) = 0$ so that $L = \frac{1}{2}mv^2$, hence

$$\begin{aligned} t - t_0 &= \pm \int_{q_0}^{q} \sqrt{\frac{m}{2E}}\, dx \\ &= \pm\sqrt{\frac{m}{2E}}(q - q_0). \end{aligned}$$

We thus recover the familiar free particle motion (exercise)

$$q = q_0 + v_0(t - t_0),$$

where

$$v_0 = \pm\sqrt{\frac{2E}{m}}$$

is the initial velocity.

Example: Constant Force

We choose $V(q) = -Fq$ where F is the constant force on the particle so that $L = \frac{1}{2}m\dot{q}^2 + Fq$. We find (exercise)

$$\begin{aligned} t - t_0 &= \pm \int_{q_0}^{q} \sqrt{\frac{m}{2(E + Fx)}}\, dx \\ &= \pm\frac{1}{F}\sqrt{2m}(\sqrt{E + Fq} - \sqrt{E + Fq_0}). \end{aligned}$$

You can easily check that this leads to the usual quadratic dependence of q on time. With $t_0 = 0$ we get

$$q = q_0 + v_0 t + \frac{1}{2} a t^2,$$

where

$$v_0 = \pm\sqrt{\frac{2}{m}(E + F q_0)}, \quad a = \frac{F}{m}.$$

Example: Harmonic Oscillator

A simple harmonic oscillator is treated via

$$\begin{aligned} t - t_0 &= \pm \int_{q_0}^{q} \sqrt{\frac{m}{2(E - \frac{1}{2}kx^2)}}\, dx \\ &= \pm\sqrt{\frac{m}{k}} \left[\sin^{-1}(\sqrt{\frac{k}{2E}} q) - \sin^{-1}(\sqrt{\frac{k}{2E}} q_0) \right]. \end{aligned}$$

We can thus write the solution in the familiar form (exercise)

$$q(t) = A \sin(\omega t + \alpha),$$

where

$$\omega = \pm\sqrt{\frac{k}{m}}$$

$$A = \sqrt{\frac{2E}{k}}$$

and

$$\alpha = \sin^{-1}(\frac{q_0}{A}).$$

More complicated potentials may or may not be amenable to analytic computations, that is, explicit evaluation of the integral. For example, the simple plane pendulum has $V(q) = mgl \cos q$ and this leads to an elliptic integral (exercise), which is not as tractable analytically as the previous examples. Still, one can extract whatever information is needed from the elliptic integral, if only numerically. Of course, in the small oscillation approximation this integral reduces to our harmonic oscillator example (see below).

Example: A Falling Rod

Let us work a non-trivial example in a little detail. Consider a thin rigid rod of mass m and length L standing upright in a uniform gravitational field. This is a state of unstable equilibrium, the slightest perturbation of its position or horizontal velocity will cause the rod to fall. Let us compute the time it takes to fall. We make the simplifying assumption that the base of the rod does not move relative to the ground. Thus the system has one degree of freedom corresponding to rotation of the rod about an axis perpendicular to and through the bottom end of the rod. We choose as our generalized coordinate the angle θ between the rod and the ground. The kinetic and potential energies are (exercise)

$$T = \frac{1}{6}mL^2\dot{\theta}^2, \quad V = \frac{mgL}{2}\sin\theta.$$

The conserved total energy is $T + V$ so that, for initial conditions characterized by θ_0 and E, we have

$$t - t_0 = \pm\int_{\theta_0}^{\theta} dx\,\sqrt{\frac{\frac{1}{3}mL^2}{2(E - \frac{mgL}{2}\sin x)}} = \pm\sqrt{\frac{L}{3g}}\int_{\theta_0}^{\theta} dx\,\sqrt{\frac{1}{\alpha - \sin x}},$$

where

$$\alpha = \frac{2E}{mgL}.$$

This results in an expression involving elliptic integrals of the second kind, which is not surprising since our model of the falling rod is just a pendulum. Let us consider the time it takes for the rod to fall to the ground. This means θ should decrease in time; we need the minus sign in the above formula. Just to get some numbers, let us choose $L = 2m$, $\alpha = 1$, $\theta = 0$, and $\theta_0 = \frac{\pi}{2} - 0.1$ radian. This corresponds to a very small initial displacement from the vertical and a very small initial velocity. We then get (exercise) $t - t_0 = 1.04s$.

Qualitative Description of the Motion

Conservation of energy determines the motion of (conservative) one-dimensional systems. But the integral relation we have derived will not, in general, lead to any simple closed form expressions such as we examined above. Nonetheless, there are still some very useful, more qualitative techniques for analyzing the motion of conservative systems with one degree of freedom using conservation of energy.

Let us note that $a(q) > 0$. This is because the kinetic energy is positive. Now, the solutions to the equations of motion must be real, it follows from our formula for the solution that the value of q must be such that total energy cannot be less than the potential energy:

$$V(q) \le E.$$

For a given value of the energy E, the isolated points where $E = V$ are called *turning points* since the velocity must vanish there. Generically, the velocity changes direction (as time increases) at a turning point, keeping the "particle" in the domain where $E \ge V(q)$. If the energy is such that the allowed region of motion is surrounded by a pair of turning points, then the motion is *bound* and periodic (exercise). If the energy is such that a given region only has one or no turning point, then the "particle" will move to infinity, perhaps after encountering a turning point. We say that the motion is *unbound*. Exceptional cases occur when the energy is such that the turning point is a maximum or minimum of the potential. In this case the motion is that of a (un)stable equilibrium or is such asymptotically.

For bound motion it is easy to see that the motion must be periodic (exercise).* The period τ of the motion can be computed from our integral expression for the solution to the equations of motion. For a given value E of the energy we have (exercise)

$$\tau(E) = \int_{q_1}^{q_2} \sqrt{\frac{2a(x)}{E - V(x)}}\, dx,$$

where the values q_1 and q_2 are the turning points of the bound motion, *i.e.,* for a given E, $V(q_1) = V(q_2) = E$. Of course, the existence of two roots is guaranteed by our assumption that the motion is bounded.†

As a simple example of this latter formula, let us return to the simple harmonic oscillator. The turning points are (exercise)

$$q_1 = -\sqrt{\frac{2E}{k}}, \quad q_2 = \sqrt{\frac{2E}{k}}.$$

We thus get (exercise)

$$\begin{aligned}
\tau(E) &= \int_{-\sqrt{\frac{2E}{k}}}^{\sqrt{\frac{2E}{k}}} \sqrt{\frac{2m}{E - \frac{1}{2}kx^2}}\, dx, \\
&= 2\sqrt{\frac{m}{k}}[\sin^{-1}(1) - \sin^{-1}(-1)] \\
&= 2\pi\sqrt{\frac{m}{k}}.
\end{aligned}$$

To summarize, for one-dimensional conservative systems the motion can be completely determined and involves the familiar cases of periodic bound motion, unbound motion, as well as stable and unstable equilibria. This very regular type of motion arises because conservation of energy is "running the show". If we drop conservation of energy, *e.g.,* we

* Note that "periodic" does not mean "harmonic". In general the period of the motion depends upon the amplitude, *i.e.,* the turning points.

† As a very nice exercise, you should be able to show that – unless the turning point is at a critical point of the potential – the period is always finite. This is not immediately obvious since the integrand blows up as you approach a turning point. What happens physically when the integral *does* diverge?

let t appear in the potential energy, then the motion of the system can be considerably more complicated. Indeed, already in the relatively simple setting of a system with one degree of freedom *and* a time dependent potential, it is possible to find chaotic motion.

Motion near equilibrium

Let us consider the motion near a stable equilibrium. Stable equilibrium occurs at points q_0 where

$$V'(q_0) = 0, \quad V''(q_0) > 0.$$

Of course, if we start the system off so that $E = V(q_0)$, then the velocity is zero and must remain zero for all time (exercise). If the system has $E - V(q_0) = \epsilon << 1$, $\epsilon > 0$, then the particle never goes very far from the equilibrium position. In this case we can approximate the potential as

$$V = V(q_0) + \frac{1}{2}V''(q_0)(q - q_0)^2$$

and we can approximate

$$a(q) \approx a(q_0).$$

With this approximation, we are back to our harmonic oscillator example with the substitutions

$$m \to a(q_0), \quad k \to V''(q_0), \quad E \to \epsilon.$$

In particular, the motion is not just periodic but harmonic (frequency independent of amplitude) with frequency

$$\omega = \sqrt{\frac{V''(q_0)}{a(q_0)}}.$$

Multi-particle systems – Example: Earth, Moon, Sun

Our final topic concerning energy is to see how to use our one particle technology to analyze multi-particle systems. This is not too difficult. The potential energy functions are split into inter-particle functions and external functions as we did with forces. To see what is going on in the context of an important class of examples, let us study the energy of the Earth-Moon and Earth-Moon-Sun system, the latter in the approximation where the Sun is fixed at the origin of our inertial reference frame.

We label the Earth as particle 1 and the Moon as particle 2. In our first model we ignore the Sun altogether.* At any given time, let the position of the Earth and Moon be

* This model is mathematically the same as you would get for any two body gravitating system, *e.g.,* a binary star system.

$\vec{r}_1$ and $\vec{r}_2$ respectively. The force on the Moon due to the Earth is

$$\vec{F}_{12} = G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2),$$

as we have seen before. The force of gravity is conservative. There will be a potential energy function $U(\vec{r}_1, \vec{r}_2))$ whose negative gradient with respect to $\vec{r}_1$ will give the force:

$$F_{12} = G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) = -\nabla U(\vec{r}_1).$$

Similarly we have $\vec{F}_{21} = -\vec{F}_{12}$ and a function $\tilde{U}(\vec{r}_1, \vec{r}_2)$ such that

$$F_{21} = G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_2 - \vec{r}_1) = -\nabla \tilde{U}(\vec{r}_2).$$

Now, notice that

$$\nabla_1 U(\vec{r}_1, \vec{r}_2) = -\nabla_2 \tilde{U}(\vec{r}_1, \vec{r}_2),$$

or more explicitly

$$\frac{\partial U}{\partial x_1} = -\frac{\partial \tilde{U}}{\partial x_2}, \quad \frac{\partial U}{\partial y_1} = -\frac{\partial \tilde{U}}{\partial y_2}, \quad \frac{\partial U}{\partial z_1} = -\frac{\partial \tilde{U}}{\partial z_2}.$$

As these relations must hold for all $\vec{r}_1$ and $\vec{r}_2$ it follows that

$$U(\vec{r}_1, \vec{r}_2) = \tilde{U}(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1 - \vec{r}_2) = -G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}.$$

Here we avoid a common abuse of notation and write $V(\vec{r}_1 - \vec{r}_2)$ instead of the more common $U(\vec{r}_1 - \vec{r}_2)$. The idea here is that the potential energy function U is a function of six variables, but is constructed via a function V of three variables. We have

$$V(x, y, z) = G\frac{m_1 m_2}{\sqrt{x^2 + y^2 + z^2}}.$$

Energy is conserved in this two particle system via two applications of the work energy theorem. As the Earth and Sun move their individual kinetic energies will change. Likewise, the potential energy U will change. But the combination

$$E = T_1 + T_2 + U = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + U(\vec{r}_1, \vec{r}_2),$$

will be unchanged. Mathematically we can see this most easily by considering the change of the system in an very small time interval dt. The energy will change as

$$dE = dT_1 + dT_2 + dU = (dT_1 + d_1 U) + (dT_2 + d_2 U) = 0 + 0.$$

Here we denoted the change in potential energy due to a small change in position of each particle as d_1U and d_2U. The zeros at the end of the equation are just coming from the work-energy theorem.

Notice that the energy of this two particle system cannot be divided up into a piece only pertaining to particle one and a piece only pertaining to particle 2. The kinetic energies *do* divide up that way, but the potential energy does not – this is the mathematical essence of the interaction between the two particles in this example.

Let us now add in the Sun, mass M with the simplifying approximation that it is fixed at the origin as an external influence. The potential energy of a particle of mass m at position $\vec{r}$ due to the sun is given by

$$U(\vec{r}) = -G\frac{Mm}{r}.$$

Thus the potential energy of the Earth-Moon system (in the presence of the Sun) now becomes

$$U(\vec{r}_1, \vec{r}_2) = -G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} - G\frac{m_1 M}{r_1} - G\frac{m_2 M}{r_2}.$$

Here we have used the fact that potential energies, like kinetic energies, are *additive* since they represent the work done by each force, which is additive.

Minus the gradient of U with respect to $\vec{r}_1$ (or $\vec{r}_2$) gives the force on the Earth (or Moon) due to the Moon (or Earth) and the Sun. It is worth taking note of the functional form of the potential energy. You note that the potential energy is of the form

$$U(\vec{r}_1, \vec{r}_2) = U_1(\vec{r}_1) + U_2(\vec{r}_2) + U_{12}(\vec{r}_1, \vec{r}_2).$$

The first two terms only refer to one of the particles and represent the external influences. The last term, which is necessarily a function of both variables, represents the interaction between the two particles.

Energy for multi-particle systems

We can easily generalize the preceding example to a system of N particles interacting with each other and their environment according to conservative forces obeying Newton's third law and the principle of superposition. We have

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \ldots + \frac{1}{2}m_N v_N^2 + U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \sum_{\alpha=1}^{N} \frac{1}{2}m_\alpha v_\alpha^2 + U.$$

where

$$\begin{aligned} U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) &= U_{12}(\vec{r}_1 - \vec{r}_2) + U_{13}(\vec{r}_1 - \vec{r}_3) + U_{23}(\vec{r}_2 - \vec{r}_3) + \ldots + U_{N-1,N}(\vec{r}_{N-1} - \vec{r}_N) \\ &\quad + U_1(\vec{r}_1) + U_2(\vec{r}_2) + \ldots + U_N(\vec{r}_N) \\ &= \sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{N} U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta) + \sum_{\alpha=1}^{N} U_\alpha(\vec{r}_\alpha). \end{aligned}$$

Here you can again see the division between terms representing interactions between the particles and terms representing interaction of the particles with their environment.

This total energy will be conserved – unchanged in time – when evaluated on any solutions $\vec{r}_\alpha(t)$ to the equations of motion,

$$\vec{F}_\alpha = m_\alpha \frac{d^2\vec{r}_\alpha(t)}{dt^2},$$

where

$$\vec{F}_\alpha = -\nabla_\alpha U.$$