Physics 4550 – Problem Set #3

The Excel worksheet Chaosgenerator.xls illustrates many important aspects of chaotic dynamics. In each of the illustrations the basic tool is the so-called Ricker Map: \( v_{n+1} = a v_n \exp(-v_n) \). The only restriction on \( a \) and \( v \) is that both are positive.

3.1 Show that the fixed points of the Ricker Map are \( v_f = 0 \) and \( v_f = \ln(a) \).

3.2 Show that the slopes of the tangent lines to the Ricker Map (i.e., \( \frac{df}{dv} \) where \( f = a \exp(-v) \)) at the respective fixed points are \( a \) and \( 1 - \ln(a) \). For what value of \( a \) does the fixed point 0 become unstable? For what value of \( a \) does the fixed point \( \ln(a) \) become unstable?

3.3 In Chaosgenerator.xls use the sheet ricker. Enter 0.99 for \( a \) in A2. Enter 10 for \( v_0 \) in A4. Click RUN. What does the time series look like? Change the 0.99 to 5. Repeat. Does the 10 in A4 matter for the time series you see? (Careful!) What is \( \ln(a) \)? How does this compare with what you see on the time series plot?

3.4 The Ricker Map undergoes a typical period doubling cascade to chaos. The bifurcation stable 1-cycle to stable 2-cycle occurs at about what value of \( a \)? (Experiment.) Successive bifurcations occur near \( a = 12.6, 14.3, \) and \( 14.7 \). Insert these values in A2 and see that. Using the last three values make a crude estimate of Feigenbaum’s number.

3.5 One of the most interesting aspects of chaos is sensitive dependence on initial conditions. This is also sometimes referred to as the “Butterfly Effect.” Sensitive dependence on initial conditions means that two time series (“trajectories”) that start close to each other eventually diverge when the system is behaving chaotically. Chaos first shows up at about \( a = 14.77 \). Use sheet butterfly. Enter 14.7 (i.e., just before chaos shows up) and two positive values in A4 and A6 that differ by less than 0.1%. The plot shows the difference between the two time series moment-by-moment. Describe what you see there. Now change A2 to 14.8 (just after chaos appears). What do you see in the plot now? Play with different starting values. (You should observe that sometimes the time series get close again. These episodes of “getting close” recur again-and-again.)

3.6 The sheet ricker2 iterates the two-dimensional system \( v_{n+1} = a v_n \exp(-v_n) \). Use \( a = 16 \) in A2 and \( a = 0.27 \) in A4, along with \( v_0 \) and \( v_1 \) both equal to 2 in A8 and A6, respectively. (The sheet defines \( y \) as the earlier \( v \) and \( x \) as the later one.) What does the first return map look like? A first return map is only a simple curve if the system producing the data is one-dimensional. In order to see a simple curve for the map above, you have to plot your data in 3-d (as \( v_n \) versus \( v_{n+1} \) versus \( v_{n+2} \)). Projected down onto a 2-d plot, the simple curve looks like its folded and overlapped. Also try 5 in each of A6 and A8. What happens in this case? The plot you saw in the first step is an attractor of the dynamics. An attractor doesn’t have to attract EVERY starting value to it. The starting values that ARE attracted to an attractor are said to lie in the attractor’s basin of attraction.

3.7 Sheet ricker3 ratchets up the dimensionality story begun in the previous question. It iterates the three-dimensional system \( v_{n+1} = a v_n \exp(-v_n) + g v_{n-1} \). Nice values of \( a \), \( g \), and \( g \) (A2, A4, A6) are 25, 0.3, and 0.3. Try starting value (A8, A10, A12) 2, 2, and 1. Compare the first return plot with what you saw in 3.6. Now change the starting values to 2, 2, and 2. Comment.