Problem Set I.a

August 28, 2014

1. a. Let a linear force on a particle of mass, $m$, always lie in the $x$-direction so that

$$\mathbf{F} = -kx \hat{i}$$

Allowing the motion to be 2-dimensional, find the motion, $\mathbf{x}(t)$, with the initial conditions ($t_0 = 0$):

$$\mathbf{x}(0) = x_0 \hat{i} + y_0 \hat{j}$$
$$\mathbf{v}(0) = v_0 \hat{j}$$

b. Let the initial conditions be

$$\mathbf{x}(0) = x_0 \hat{i} + y_0 \hat{j}$$
$$\mathbf{v}(0) = v_{0x} \hat{i} + v_{0y} \hat{j}$$

Prove that the solutions are equivalent up to a different choice of the initial time $t_0$ and a specific relationship between $v_0$ and $(v_{0x}, v_{0y})$.

2. Two dimensional oscillator

Now consider a radial Hooke’s law force in 2-dimensions,

$$\mathbf{F} = -kr \hat{r}$$

where the force is along the radial unit vector $\hat{r}$ and depends on the distance from the origin, $r$, where

$$\hat{r} = \hat{i} \cos \varphi + \hat{j} \sin \varphi$$
$$r = \sqrt{x^2 + y^2}$$

and therefore

$$\mathbf{r} = r \hat{r} = x \hat{i} + y \hat{j}$$

Let the initial position and velocity (at $t_0 = 0$) be

$$\mathbf{x}(0) = x_0 \hat{i}$$
$$\mathbf{v}(0) = v_0 \hat{j}$$

Find the motion, $\mathbf{x}(t)$.

3. Solve for the motion of a block of mass, $m$, down an inclined plane of angle $\theta$, which is connected by a rope to a rolling cart of mass $M$. Neglect the moment of inertia of the wheels of the cart and the pulley. Assume friction between the block and the inclined plane of

$$\mathbf{f} = -\mu \mathbf{N}$$
where $N$ is the normal force exerted by the plane on the block.

4. We have proved that the angular momentum,

$$L = (r(t) - R) \times p(t)$$

of a single particle about a fixed position, $R$, is related to the torque,

$$N = (r(t) - R) \times F(t)$$

by the equation

$$N = \frac{dL}{dt}$$

Now consider an isolated system of particles with positions $r_i$ and momenta $p_i$ where the force on the $i^{th}$ particle by the $j^{th}$ particle is $F_{ij}$. Assume Newton’s 3rd Law holds, so that

$$F_{ji} = -F_{ij}$$
$$F_{ii} = 0$$

Assume that the force between any two of the particles is a central force, i.e., it lies along the vector, $r_i(t) - r_j(t)$, between the two particles.

The angular momentum of the $i^{th}$ particle is then

$$L_i = (r_i(t) - R) \times p_i(t)$$

with the torque on the $i^{th}$ particle produced by the $j^{th}$ given by

$$N_{ij} = (r_i(t) - R) \times F_{ij}(t)$$

The total torque on the $i^{th}$ particle

$$N_i = \sum_{j=1}^{N} (r_i(t) - R) \times F_{ij}(t)$$

therefore satisfies,

$$N_i = \frac{dL_i}{dt}$$

Show that the total angular momentum of the system,

$$L_{tot} = \sum_{i=1}^{N} L_i$$

is conserved.