1 Solving the equation of motion

We have shown that the action for any two body system acted on by a central force may be written as

\[
S = \int_0^t \left( \frac{1}{2} \mu \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r) \right) dt
\]

where \( \mu = \frac{mM}{M+m} \) is the reduced mass and \( L = \mu r^2 \dot{\phi} \) the conserved angular momentum.

The equation of motion was found to be

\[
\mu \ddot{r} - L^2 \mu r^3 + \frac{\partial V}{\partial r} = 0
\]

but we work instead with the conserved energy,

\[
E = \frac{1}{2} \mu \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) + V(r)
\]

\[
= \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2 \mu r^2} + V(r)
\]

Notice that we may have \( E < 0 \). The energy is fixed by its initial value. Taking \( r = r_{\text{min}} \) for a bounded orbit at \( t = 0 \),

\[
E = \frac{L^2}{2 \mu r_{\text{min}}^2} + V(r_{\text{min}})
\]

1.1 Additional conserved quantities

From the angular momentum and the energy we may construct another conserved quantity. The time rate of change of the unit vector \( \hat{\phi} \) is given by

\[
\frac{d}{dt} \hat{\phi} = \frac{d}{dt} \left( -i \sin \phi + j \cos \phi \right) = -i \cos \phi - j \sin \phi \dot{\phi} = -\dot{\phi} \hat{r}
\]

and therefore, using \( L = \mu r^2 \dot{\phi} \), we have

\[
\frac{d}{dt} \dot{\phi} = -\frac{L}{\mu r^2} \hat{r} = \frac{L}{\mu \alpha} F = \frac{d}{dt} \left( \frac{L}{\mu \alpha} P \right)
\]
where the force is given by \( F = -\frac{GMm}{r^2} \hat{r} \equiv -\frac{\alpha}{r^2} \hat{r} \) and we have

\[
\frac{d}{dt} \left( p - \frac{\mu \alpha}{L} \dot{\phi} \right) = 0
\]

Therefore, Hamilton’s vector,

\[
h = p - \frac{\mu \alpha}{L} \dot{\phi}
\]

is conserved as a consequence of rotational invariance.

Since angular momentum is conserved, the product

\[
A = h \times L = \left( p - \frac{\mu \alpha}{L} \dot{\phi} \right) \times L
\]

as also conserved. This is the Laplace-Runge-Lenz vector.

### 1.2 Solving using Hamilton’s vector

Choose the initial conditions so that at time \( t = 0 \) the particle lies at perihelion, \( r_{\text{min}} = b \), at \( \phi = 0 \). This is a turning point, so \( \dot{r} = 0 \) and

\[
v = v_0 \hat{j} = \mu b \dot{\phi}_0 \hat{j}
\]

Then Hamilton’s vector is

\[
h = p - \frac{\mu \alpha}{L} \dot{\phi} = \left( \mu b \dot{\phi}_0 - \frac{\mu \alpha}{L} \right) \hat{j}
\]

At any later time,

\[
(h \cdot \dot{\phi})_{\text{initial}} = h \cdot \dot{\phi} = \left( \mu b \dot{\phi}_0 - \frac{\mu \alpha}{L} \right) \hat{j} \cdot \dot{\phi} = \left( p - \frac{\mu \alpha}{L} \dot{\phi} \right) \cdot \dot{\phi} = \mu r \dot{\phi} - \frac{\mu \alpha}{L}
\]

\[
\left( \mu b \dot{\phi}_0 - \frac{\mu \alpha}{L} \right) \cos \phi = \mu r \dot{\phi} - \frac{\mu \alpha}{L}
\]

\[
\left( \frac{L}{b} - \frac{\mu \alpha}{L} \right) \cos \phi = \frac{L}{r} - \frac{\mu \alpha}{L}
\]

and therefore

\[
r = \frac{L}{\mu \alpha} + \left( \frac{L}{b} - \frac{\mu \alpha}{L} \right) \cos \phi
\]

\[
r = \frac{L^2}{\mu \alpha} \frac{1}{1 + \left( \frac{L^2}{\mu \alpha b} - 1 \right) \cos \phi}
\]
1.3 Fitting the constants

So far, our solution is expressed in terms of constants $L$ and $r_0$. It is convenient to define

$$r_0 \equiv \frac{L^2}{\mu \alpha}$$

$$\varepsilon \equiv \frac{L^2}{\mu \alpha b} - 1$$

so that the orbit equation takes the simpler form

$$r = \frac{r_0}{1 + \varepsilon \cos \varphi}$$

Then at $\varphi = 0$ and $\varphi = \pi$, $r$ lies along the $x$ axis, so the length of the semimajor axis is

$$2a = \frac{r_0}{1 + \varepsilon} + \frac{r_0}{1 - \varepsilon} = \frac{r_0 (1 - \varepsilon) + r_0 (1 + \varepsilon)}{1 - \varepsilon^2} = \frac{2r_0}{1 - \varepsilon^2}$$

$$a = \frac{r_0}{1 - \varepsilon^2}$$

Along the $y$ axis we have the *semi latus rectum*, that is, the distance to the ellipse from the center of force at the focus, perpendicular to the major axis,

$$2p = \frac{r_0}{1 + \varepsilon \cos \frac{\pi}{2}} + \frac{r_0}{1 + \varepsilon \cos \frac{3\pi}{2}}$$

$$2p = 2r_0$$

$$p = r_0$$

For $\varepsilon < 1$ we have $p$. The semiminor axis has length $b$ equal to the maximum $y$-coordinate, where $y = r \sin \varphi$.

Thus

$$y = \frac{p \sin \varphi}{1 + \varepsilon \cos \varphi}$$

$$0 = \frac{dy}{d\varphi}$$

$$= \frac{p \cos \varphi}{1 + \varepsilon \cos \varphi} - \frac{-\varepsilon p \sin^2 \varphi}{(1 + \varepsilon \cos \varphi)^2}$$

$$= \frac{p \cos \varphi (1 + \varepsilon \cos \varphi) + \varepsilon p \sin^2 \varphi}{(1 + \varepsilon \cos \varphi)^2}$$

$$= \frac{p \cos \varphi + \varepsilon p \cos^2 \varphi + \varepsilon p \sin^2 \varphi}{(1 + \varepsilon \cos \varphi)^2}$$

Then

$$\cos \varphi_m = -\varepsilon$$

and therefore,

$$b = \frac{p \sin \varphi_m}{1 + \varepsilon \cos \varphi_m}$$
The energy is

\[ E = \frac{L^2}{2\mu b^2} + V(b) \]

\[ = \frac{L^2}{2\mu b^2} - \frac{\alpha}{b} \]

Therefore,

\[ \frac{L^2}{2\mu b^2} - \frac{\alpha}{b} - E = 0 \]

\[ \frac{1}{b} = \frac{\alpha \pm \sqrt{\alpha^2 + \frac{2EL^2}{\mu}}}{L^2/\mu} \]

\[ = \frac{\alpha\mu}{L^2} \left( 1 + \sqrt{1 + \frac{2EL^2}{\alpha^2\mu}} \right) \]

Therefore,

\[ \varepsilon = \frac{L^2}{\mu\alpha b} - 1 \]

\[ = \sqrt{1 + \frac{2EL^2}{\alpha^2\mu}} \]

and we have the solution in terms of energy and angular momentum,

\[ r = \frac{p}{1 + \varepsilon \cos \varphi} \]

\[ \varepsilon = \sqrt{1 + \frac{2Ep}{\alpha}} \]

\[ p = \frac{L^2}{\mu\alpha} \]

\[ a = \frac{p}{1 - \varepsilon^2} \]

\[ b = a\sqrt{1 - \varepsilon^2} \]