General coordinate covariance of the Euler Lagrange equations

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Here we show that the Euler-Lagrange equation is covariant under general coordinate transformations. By this we mean that if the Euler-Lagrange equation

\[ V_i (x) \equiv \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \]

is satisfied in one set of coordinates, \( x^i \), then it will hold in any other, \( y^i \),

\[ V_i (y) \equiv \frac{d}{dt} \frac{\partial L}{\partial \dot{y}_i} - \frac{\partial L}{\partial y_i} = 0 \]

where \( x^i (y^j) \) is the invertible coordinate transformation. For the two vectors to vanish together requires there to be a linear map from one to other, i.e., there exists some \( J^j_i \) such that

\[ V_i = \sum_j J^j_i V_j \]

It is clear what \( J^j_i \) must be – if \( L \) is independent of velocity, we require

\[ \frac{\partial L}{\partial x_i} = \sum_j J^j_i \frac{\partial L}{\partial y_j} \]

but the chain rule tells us that

\[ \frac{\partial L}{\partial x_i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial L}{\partial y^j} \]

Therefore, \( J^j_i \) is the Jacobian matrix of the coordinate transformation, \( \frac{\partial y^i}{\partial x^j} \). In conclusion, the Euler-Lagrangian equation hold in any coordinate system if and only if

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \sum_j \frac{\partial y^j}{\partial x^i} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{y}_j} - \frac{\partial L}{\partial y_j} \right) \]

for any two, \( x^i \rightarrow y^i \).

We prove that this is the case by deriving the relationship between the Euler-Lagrange equation for \( x^i (t) \) and the Euler-Lagrange equation for \( y^i (t) \). Consider the variational equation for \( y^i \), computed in two ways. Since the action may be written as either \( S [x^i] \) or \( S [y^i] \), we have

\[ S [y^i] = S [x^i (y^k)] \]
First, we may immediately write the Euler-Lagrange equation by varying $S[y^i(t)]$. Following the usual steps, integrating by parts, we have

$$
\delta S = \int_C L(y^i, \dot{y}^i, t) \, dt
$$

$$
= \sum_{k=1}^N \int_C \left( \frac{\partial L}{\partial y^k} \delta y^k + \frac{\partial L}{\partial \dot{y}^k} \dot{y}^k \right) \, dt
$$

$$
= \sum_{k=1}^N \int_C \left( \frac{\partial L}{\partial y^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}^k} \right) \right) \delta y^k \, dt
$$

where the surface term vanishes in the final step because the variation is taken to vanish at the endpoints.

Now compare what we get by varying $S[x^i(y^k)]$ with respect to $y^i(t)$:

$$
0 = \delta S
$$

$$
= \delta \int_C L(x^i(y^k, t), \dot{x}^i(y^k, \dot{y}^k, t)) \, dt
$$

$$
= \sum_{i,k=1}^N \int_C \left( \frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial y^i} \delta y^i + \frac{\partial L}{\partial \dot{x}^k} \left( \frac{d}{dt} \frac{\partial x^k}{\partial y^i} \right) \delta y^i \right) \, dt
$$

Since $x^i$ is a function of $y^k$ and $t$ only, $\frac{\partial x^k}{\partial y^i} = 0$ and the second term in the first parentheses vanishes.

Now we need two identities. Explicitly expanding the velocity, $\dot{x}^k$, the chain rule gives:

$$
\dot{x}^k = \frac{dx^k}{dt}
$$

$$
= \frac{d}{dt} x^k (y^i(t), t)
$$

$$
= \frac{\partial x^k}{\partial y^i} \dot{y}^i + \frac{\partial x^k}{\partial t}
$$

so differentiating, we have one identity,

$$
\frac{\partial \dot{x}^k}{\partial y^i} = \frac{\partial x^k}{\partial y^i}
$$

For the second identity, we differentiate eq.(1) for the velocity with respect to $y^i$:

$$
\frac{\partial \dot{x}^k}{\partial y^i} = \frac{\partial^2 x^k}{\partial y^i \partial y^i} \dot{y}^i + \frac{\partial^2 x^k}{\partial y^i \partial t}
$$

$$
= \frac{\partial}{\partial y^i} \left( \frac{\partial x^k}{\partial y^i} \right) \dot{y}^i + \frac{\partial}{\partial t} \left( \frac{\partial x^k}{\partial y^i} \right)
$$

$$
= \frac{d}{dt} \left( \frac{\partial x^k}{\partial y^i} \right)
$$

Now return and substitute into the variation

$$
0 = \delta S
$$

$$
= \sum_{i,k=1}^N \int_C \left( \frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial y^i} \delta y^i + \frac{\partial L}{\partial \dot{x}^k} \left( \frac{d}{dt} \frac{\partial x^k}{\partial y^i} \delta y^i \right) \right) \, dt
$$

$$
= \sum_{i,k=1}^N \int_C \left( \frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial y^i} \delta y^i + \frac{\partial L}{\partial \dot{x}^k} \left( \frac{d}{dt} \frac{\partial x^k}{\partial y^i} \delta y^i \right) \right) \, dt
$$

$$
= \sum_{i,k=1}^N \int_C \left( \frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial y^i} \delta y^i + \frac{\partial L}{\partial \dot{x}^k} \frac{d}{dt} \left( \frac{\partial x^k}{\partial y^i} \delta y^i \right) \right) \, dt
$$

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Finally, integrate the final term by parts,

$$\sum_{i,k=1}^{N} \int C \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \delta y^i \right) dt = \sum_{i,k=1}^{N} \int C \left( \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial y^i} \delta y^i \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \right) \frac{\partial x^k}{\partial y^i} \delta y^i \right) dt$$

$$= \sum_{i,k=1}^{N} \left( \frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial y^i} \delta y^i \right)_{\text{final}} - \left( \frac{\partial L}{\partial x^k} \frac{\partial x^k}{\partial y^i} \delta y^i \right)_{\text{initial}} - \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \right) \frac{\partial x^k}{\partial y^i} \delta y^i \right) dt$$

$$= \sum_{i,k=1}^{N} \left( \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \right) \frac{\partial x^k}{\partial y^i} \delta y^i \right) dt$$

where $\delta y_i$ vanishes at the endpoints. The vanishing variation now becomes

$$0 = \sum_{i,k=1}^{N} \left( \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \right) \frac{\partial x^k}{\partial y^i} \delta y^i \right) dt$$

The initial equality of the two forms of the action, $S[y^i] = S[x^i(y^k)]$ implies $\delta S[y^i] = \delta S[x^i(y^k)]$ and therefore

$$\sum_{k=1}^{N} \int C \left( \frac{\partial L}{\partial y^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^k} \right) \right) \delta y^k dt - \sum_{i,k=1}^{N} \int C \left( \frac{\partial L}{\partial x^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \right) \right) \frac{\partial x^k}{\partial y^i} \delta y^i dt = 0$$

$$\sum_{k=1}^{N} \int C \left[ \left( \frac{\partial L}{\partial y^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^k} \right) \right) - \left( \frac{\partial L}{\partial x^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \right) \right) \frac{\partial x^k}{\partial y^i} \right] \delta y^i dt = 0$$

and the independence and arbitrariness of the variation, $\delta y^i$ implies covariance:

$$\left( \frac{\partial L}{\partial y^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^k} \right) \right) = \left( \frac{\partial L}{\partial x^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial x^k} \right) \right) \frac{\partial x^k}{\partial y^i}$$

The conclusion we reach is that no matter what coordinates $q^i$ we choose for a problem, we may always write the equation of motion as

$$\frac{\partial L}{\partial q^k} - \frac{d}{dt} \left( \frac{\partial L}{\partial q^k} \right) = 0$$

The same is true of the action. Rather than writing the Euler-Lagrange equation, we may write the action as the integral of the Lagrangian and write the Lagrangian in terms of whatever coordinates we choose,

$$S[q^i] = \int_{t_1}^{t_2} L(q^i, \dot{q}^i, t) dt$$