Lecture 44

Relevant sections in text: §

The Hamiltonian of the electromagnetic field (cont.)

So far, we have expressed the electromagnetic Hamiltonian as the following function of the Fourier components of the vector potential in the radiation gauge

$$H = \int d^3k \left\{ \frac{1}{c^2} |\dot{\mathbf{A}}_{\mathbf{k}}|^2 + k^2 |A_{\mathbf{k}}|^2 \right\}.$$

Thus, you can think of the vector potential as a sort of generalized coordinate, and the electric field as the canonically conjugate momentum. The fact that there is a continuous family of coordinates and momenta (one for each spatial point, or one for each wave vector) leads to the statement that the EM field has "an infinite number of degrees of freedom". This is the principal feature distinguishing quantum field theory from quantum mechanics, and it is what allows one to describe processes in which particles are created and destroyed.

Now let us put our formula for the Hamiltonian into an even nicer form. We do this by taking account of the properties of the Fourier components of the vector potential. We have seen that the Fourier component with wave vector $\mathbf{k}$ of the vector potential is orthogonal to $\mathbf{k}$, and satisfy some complex conjugation relation. We take these into account (and introduce a convenient normalization) via the definition

$$\mathbf{A}_{\mathbf{k}} = \sum_{\sigma=1}^{2} \sqrt{\frac{\hbar c}{k}} (a_{\mathbf{k},\sigma} \epsilon_{\mathbf{k},\sigma} + a^*_{-\mathbf{k},\sigma} \epsilon_{-\mathbf{k},\sigma}),$$

where σ labels the polarization and $\epsilon_{\mathbf{k},\sigma}$ are two orthonormal vectors orthogonal to $\mathbf{k}$. The variables $a_{\mathbf{k},\sigma}$ carry the amplitude information about each polarization, and the polarization direction is determined by the choice of $\epsilon_{\mathbf{k},\sigma}$. Note that we have used $\hbar$ to define the new variables. From a purely classical EM point of view this is a bit strange, but it is convenient for the quantum treatment we will give. For now, just think of the use of $\hbar$ as a way of giving convenient units to the amplitudes $a_{\mathbf{k},\sigma}$, which are dimensionless (exercise).

Next, we consider the remaining Maxwell equations,

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0$$

and

$$\nabla \cdot \mathbf{E} = 0.$$

These are the "equations of motion" for the vector potential, or equivalently, the amplitudes $a_{\mathbf{k},\sigma}$. They imply that each component of $\mathbf{A}$ satisfies the wave equation with propagation velocity c (to this we must adjoin the side condition $\nabla \cdot \mathbf{A} = 0$). This means that (exercise)

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{c}_{\mathbf{k}} e^{i\omega(k)t} + \mathbf{c}^*_{-\mathbf{k}} e^{-i\omega(k)t},$$

for some constants $\mathbf{c}_{\mathbf{k}}$. Using these equations and the form of $\mathbf{A}_{\mathbf{k}}(t)$ given above it follows that

$$\frac{d}{dt}\mathbf{A}_{\mathbf{k}} = i\sqrt{\hbar k c}\sum_{\sigma=1}^{2}(a_{\mathbf{k},\sigma}\epsilon_{\mathbf{k},\sigma} - a^*_{-\mathbf{k},\sigma}\epsilon_{-\mathbf{k},\sigma}).$$

All together, we have (exercise)

$$H = \int d^3k \sum_{\sigma} \hbar\omega(k)\left(a^*_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right),$$

where

$$\omega(k) = kc.$$

Hopefully, this very simple form for the energy justifies to you all the effort that went into deriving it. Up to a choice of zero point of energy, this is clearly a classical version of the Hamiltonian for a collection of oscillators labeled by $\mathbf{k}$ and σ. Thus, mathematically at least, the electromagnetic field (in the radiation gauge) *is* a continuous collection of harmonic oscillators.*

The quantization of the EM field

The classical Hamiltonian for the electromagnetic field can be expressed as a continuous superposition over harmonic oscillator Hamiltonians:

$$H = \int d^3k \sum_{\sigma} \hbar\omega(k)\left(a^*_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right).$$

We thus view the quantum EM field as an infinite set of quantum oscillators. The oscillators' degrees of freedom are labeled by the wave vector $\mathbf{k}$ and the polarization σ. We view the ladder operators for each degree of freedom as $a_{\mathbf{k},\sigma}$ and $a^\dagger_{\mathbf{k},\sigma}$. In the context of quantum field theory, we call these operators *annihilation* and *creation* operators, respectively. We will see why in a moment. The annihilation and creation operators satisfy the commutation relations

$$[a_{\mathbf{k},\sigma}, a^\dagger_{\mathbf{k}',\sigma'}] = \delta(\mathbf{k}-\mathbf{k}')\delta_{\sigma,\sigma'}I.$$

The quantum Hamiltonian is built from the creation and annihilation operators via

$$H = \int d^3k \sum_{\sigma} \hbar\omega(k)\left(a^\dagger_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right).$$

* A similar computation with the Lagrangian for the Maxwell field,

$$L = \frac{1}{8\pi}\int_{\text{all space}} d^3x(E^2 - B^2),$$

leads to a sum (really, integral) of harmonic oscillator Lagrangians.

This is clearly just the sum of energies for each individual oscillator (with the "zero point energy" dropped).

Incidentally, it is not too hard to compute the total momentum $\mathbf{P}$ of the electromagnetic field. It is obtained from the integral of the Poynting vector. (This means that the Cartesian components of the total momentum are integrals of the corresponding components of the Poynting vector field.) In terms of the creation and annihilation operators we get (exercise)

$$\mathbf{P} = \frac{c}{4\pi}\int d^3x\, \mathbf{E}\times\mathbf{B} = \int d^3k \sum_\sigma \hbar\mathbf{k}\left(a^\dagger_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right).$$

This is the same as the total energy except that the energy of each mode, $\hbar\omega(k)$ has been replaced by the momentum of each mode $\hbar\mathbf{k}$. Using the commutation relations between annihilation and creation operators it follows that the energy and momentum are compatible observables. This can be interpreted as saying (i) the momentum is conserved, (ii) the energy is translation invariant.

Do you recall the usual lore of photons? You know, the lore that says a photon with a definite energy and momentum will have $E = \hbar\omega$, and $\mathbf{P} = \hbar\mathbf{k}$, where $\omega = kc$. We see that we are in a position to model a photon with definite energy and momentum as a quantum normal mode of the EM field!

To make detailed sense of all this we should spell out the meaning of all these ladder operators. The idea is to first use the harmonic oscillator point of view to understand the definition of the operators. Then we can reinterpret the mathematical set-up in terms of photon states.

The vacuum

To begin with, let us suppose that all the oscillators are in their ground state. This state of the EM field, denoted $|0\rangle$, is called the *vacuum state*. You can easily see why. This state is an eigenstate of H and $\mathbf{P}$ with eigenvalue zero:

$$H|0\rangle = 0 = \mathbf{P}|0\rangle.$$

This you can verify from the fact that all the annihilation (*i.e.*, lowering) operators map the ground state to the zero vector:

$$a_{\mathbf{k},\sigma}|0\rangle = 0.$$

Thus the vacuum of the EM field (in the absence of any other interactions, which we are not modeling here) is a state in which the energy and momentum are known with probability one. It can be shown that this state is the state of lowest energy and momentum. Note, in particular, that the vacuum is a stationary state.

By the way, have you ever encountered claims that, thanks to the uncertainty principle, there is an "infinite reservoir of zero point energy in the vacuum"? Perhaps you have even seen seemingly learned schemes to extract this energy for practical use. Now you are in a position to be a little skeptical: the total energy-momentum of the vacuum is perfectly well-defined – it vanishes! Where do these wacky claims come from? As with most popular distortions of scientific results, there is a kernel of truth here. To uncover it, return to the case of a single harmonic oscillator. The ground state energy is perfectly well defined, but the energy and position and/or momentum operators are not compatible. This means that if you know the energy with probability one, you cannot know the position or momentum of the oscillator with probability one. Indeed, we saw that in the ground state of the oscillator the probability distributions for position and momentum are Gaussians with zero average. The EM field has a similar behavior. Recall that the "position" is, essentially, the vector potential (hence the magnetic field is a "function of position") and the "momentum" is, essentially, the electric field. In the EM vacuum state, the energy is minimized and sharply defined, but the EM fields themselves "fluctuate". More precisely, the EM fields have a probability distribution with non-zero standard deviation; basically, each mode (and polarization) is a described by a Gaussian probability distribution in the vacuum state. (This fact is responsible for things like the "Casimir effect" and the "Lamb shitf". It is also can be viewed as responsible for spontaneous emission as we shall see.) Very roughly speaking, the uncertainty principle means that, while the total energy is known with certainty, the energy density is "uncertain". I say "very roughly" since the notion of quantum energy density of an EM field is rather touchy; there is no well-defined quantum version that behaves much like the classical analog. Indeed, much of the zero point energy nonsense that appears in print is based upon trying to use classical ideas to describe a feature of the theory that is, well, not at all classical!

Photon states

With the vacuum state under control, we can now consider *excited stationary states of the quantum EM field*, which we will interpret as states with one or more photons. Think again about the EM field as a large collection of harmonic oscillators, labeled by wave number and polarization. Suppose we put one of these oscillators in its first excited state, say, by applying $a^\dagger_{\mathbf{k},\sigma}$ to the vacuum for some fixed choice of $\mathbf{k}$ and σ. We denote this state as

$$|1_{\mathbf{k},\sigma}\rangle = a^\dagger_{\mathbf{k},\sigma}|0\rangle.$$

Using the commutation relations between the annihilation and creation operators it can be shown that the resulting state is a stationary state, in fact an eigenvector of both H and $\mathbf{P}$:

$$H|1_{\mathbf{k},\sigma}\rangle = \hbar\omega(k)|1_{\mathbf{k},\sigma}\rangle,$$

$$\mathbf{P}|1_{\mathbf{k},\sigma}\rangle = \hbar\mathbf{k}|1_{\mathbf{k},\sigma}\rangle.$$

This state has energy-momentum values defined with probability unity, which take the form appropriate for a single photon. We interpret this state as a *1 photon state* with the indicated wave number, frequency, and polarization. Thus, the first excited stationary state of the quantum electromagnetic field can be viewed as a single photon. In this sense photons are "quanta of the electromagnetic field". Superpositions of photon states over momenta lead to photons that have probability distributions for their energy and momenta. Thus photons need not have a definite momentum or energy any more than, say, an electron must.

More generally, we can build up an infinite set of states by repeatedly applying to the vacuum state the creation operators with various wave numbers and polarizations. The result of each application of the creation operator $a^\dagger_{\mathbf{k},\sigma}$ is to define a state with one more photon of the indicated type. In this way one can build a basis of energy eigenvectors. Each application of the annihilation operator $a_{\mathbf{k},\sigma}$ results in a state with a photon of the indicated type removed. If the state doesn't have such a photon, (*e.g.,* the vacuum state won't) then the result is the zero vector.

One can of course take superpositions of the stationary states; such states need not have a simple particle interpretation, but instead can be interpreted as having a probability distribution for encountering photons of various types. Of course, such states will not be stationary states.

We see, then, how the "normal modes" of a field satisfying wave-type equations can be "quantized". The resulting theory admits particle-like properties in its stationary states. To date, every known elementary particle has been successfully described by a quantum field in much the way we described photons using a quantized electromagnetic field. Interactions between particles (better: between quantum fields) have also been described with considerable success using the quantum field formalism. Indeed, it is reasonable to adopt the point of view that, in our current best understanding of nature, quantum fields are the stuff out of which everything is made!

Photons coupled to a charge

We are now ready to define the quantum system consisting of a charged particle (mass m, charge q) coupled to the electromagnetic field (*i.e.,* photons). The Hilbert space is the tensor product of the usual Hilbert space of a (spinless, for simplicity) particle and the photon Hilbert space we have just described. If we assume the electron is bound to a nucleus via a potential V, then the Hamiltonian is

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + V(\vec{X}) + \frac{1}{8\pi}\int d^3x\,(E^2 + B^2).$$

This looks a lot like our usual (semi-classical) form of the Hamiltonian, with the addition of the electromagnetic energy operator, which can be expressed in terms of annihilation and creation operators as detailed earlier. There are some important new ingredients however. First of all this Hamiltonian is meant to act on the tensor product space defined above. Secondly, in our previous quantum mechanical treatment of the electromagnetic interaction we simple assumed $\vec{A}$ was some given function of $\vec{x}$ and t and we turned it into an operator on the Hilbert space for a particle by sending $\vec{x} \to \vec{X}$. Now we use the general Fourier form of the vector potential as described earlier. We still send $\vec{x} \to \vec{X}$ in that formula, but the vector potential is now a non-trivial operator on the photon part of the Hilbert space as well, since its Fourier components are linear combinations of the creation and annihilation operators. Thus the kinetic energy part of the Hamiltonian is a rather complicated combination of particle and photon operators.*

* As far as I know, it is an open question how to give a mathematically rigorous definition to the resulting kinetic energy operator. So far one can only make sense of it in the context of perturbation theory.