

Lecture 32

Relevant sections in text: §3.7, 3.9

Total Angular Momentum Eigenvectors

How are the total angular momentum eigenvectors related to the original product eigenvectors (eigenvectors of L_z and S_z)? We will sketch the construction and give the results. Keep in mind that the general theory of angular momentum tells us that, for a given l , there are only two values for j , namely, $j = l \pm \frac{1}{2}$.

Begin with the eigenvectors of total angular momentum with the maximum value of angular momentum:

$$|j_1 = l, j_2 = \frac{1}{2}, j = l + \frac{1}{2}, m_j = l + \frac{1}{2}\rangle.$$

Given j_1 and j_2 , there is only one linearly independent eigenvector which has the appropriate eigenvalue for J_z , namely

$$|j_1 = l, j_2 = \frac{1}{2}, m_1 = l, m_2 = \frac{1}{2}\rangle,$$

so we have

$$|j_1 = l, j_2 = \frac{1}{2}, j = l + \frac{1}{2}, m = l + \frac{1}{2}\rangle = |j_1 = l, j_2 = \frac{1}{2}, m_1 = l, m_2 = \frac{1}{2}\rangle.$$

To get the eigenvectors with lower eigenvalues of J_z we can apply the ladder operator

$$J_- = L_- + S_-$$

to this ket and normalize it. In this way we get expressions for all the kets

$$|j_1 = l, j_2 = 1/2, j = l + \frac{1}{2}, m_j\rangle, \quad m_j = -l - \frac{1}{2}, -l + \frac{1}{2}, \dots, l - \frac{1}{2}, l + \frac{1}{2}.$$

The details can be found in the text.

Next, we consider $j = l - \frac{1}{2}$ for which the maximum m_j value is $m_j = l - \frac{1}{2}$. This eigenvalue is doubly degenerate: there is a 2-d space with this value for m_j . A basis for this space is provided by the product vectors:

$$|j_1 = l, j_2 = \frac{1}{2}, m_1 = l, m_2 = -\frac{1}{2}\rangle, \quad |j_1 = l, j_2 = \frac{1}{2}, m_1 = l - 1, m_2 = \frac{1}{2}\rangle$$

Of course, two of the *total* angular momentum eigenvectors (which we are trying to compute) will also provide a basis for this space. One of the basis vectors is already known; it has $j = l + \frac{1}{2}$, $m_j = l - \frac{1}{2}$ and its computation was explained in the previous paragraph. The eigenvector we are after (with $j = l - 1/2$, $m_j = l - 1/2$) can now be computed by

demanding it be a linear combination of the above product vectors which is orthogonal to the already known eigenvector with $j = l + \frac{1}{2}$, $m_j = l - \frac{1}{2}$. With the vector characterized by $j = l - \frac{1}{2}$, $m_j = l - \frac{1}{2}$ in hand one can use the ladder operator to get all the other m_j values, that is we will obtain

$$|j_1 = l, j_2 = 1/2, j = l - \frac{1}{2}, m_j\rangle, \quad m_j = -l + \frac{1}{2}, -l + \frac{3}{2}, \dots, l - \frac{3}{2}, l - \frac{1}{2}.$$

In this way we can express all the total angular momentum eigenvectors in terms of the product eigenvectors.

The result of all these considerations is not too complicated. We have, for $j = l \pm \frac{1}{2}$,

$$|j_1 = l, j_2 = \frac{1}{2}, j = l \pm \frac{1}{2}, m_j\rangle = \frac{1}{\sqrt{2l+1}} \left\{ \pm \sqrt{l \pm m_j + \frac{1}{2}} |l, m_j - \frac{1}{2}\rangle \otimes |+\rangle + \sqrt{l \mp m_j + \frac{1}{2}} |l, m_j + \frac{1}{2}\rangle \otimes |-\rangle \right\}.$$

In terms of spinor fields we define the *total angular momentum spinors* representing the total angular momentum eigenvectors, by taking components in the product basis:

$$\Psi_{j,m} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm \sqrt{l \pm m_j + \frac{1}{2}} a_+(r) Y_{l, m_l = m_j - \frac{1}{2}}(\theta, \phi) \\ \sqrt{l \mp m_j + \frac{1}{2}} a_-(r) Y_{l, m_l = m_j + \frac{1}{2}}(\theta, \phi) \end{pmatrix},$$

The functions $a_{\pm}(r)$ are not determined by the angular momentum information. The upper (lower) component of this column vector determines the probability density for finding a particle at the position specified by (r, θ, ϕ) and with a spin up (down) along z , given that its total angular momentum is known with certainty to have the values specified by $j = l \pm \frac{1}{2}$ and m_j .

Spin correlations and quantum weirdness: Spin 1/2 systems

The principal difference between quantum mechanics and classical mechanics is the incompatibility of certain observables. This incompatibility can lead to some dramatic results. Recall the results of adding two spin 1/2 angular momenta. There the total spin observables are not all compatible with the individual spin observables. This leads to some pretty weird stuff from a classical physics point of view. A key issue was raised by Einstein-Podolsky and Rosen (EPR) in a famous critique of the completeness of quantum mechanics. Much later, Bell showed that the basic ‘‘classical locality’’ assumption of EPR must be violated, implying in effect that nature is truly as weird as quantum mechanics makes it out to be. Here we give a brief discussion of some of these ideas.

Consider a pair of spin 1/2 particles created in a spin singlet state. (Experimentally speaking, this can be done in a variety of ways; see your text.) Thus the spin state of the system is defined by the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).$$

Let us suppose that the particles propagate – undisturbed and not interacting – until they are well-separated. We have two observers. Observer 1 is equipped to measure the spin properties of Particle 1 and Observer 2 is equipped to measure the spin properties of particle 2. To begin, suppose both observers measure spin component along the z axis. If observer 1 sees spin up, what does observer 2 see? You probably will correctly guess: spin down. But can you prove it? Well, the reason for this result is that the state of the system is an eigenvector of \mathbf{S}_z with eigenvalue zero. So, the two particles are known with certainty to have opposite values for their z -components of spin. Alternatively, you can see from the expansion of $|\psi\rangle$ in the product basis that the only states that occur (with equal probability) are states with opposite spins. Let us see how to prove this systematically; it's a good exercise.

We can ask the question as the following sequence of simple questions. What is the probability $P(S_{1z} = \frac{\hbar}{2})$ that observer 1 gets spin up? That's easy:*

$$P(S_{1z} = \frac{\hbar}{2}) = |\langle ++|\psi\rangle|^2 + |\langle +-|\psi\rangle|^2 = \frac{1}{2}.$$

Of course, there is nothing special about particle 1 compared to particle 2; the same result applies to particle 2. What's the probability for getting particle 1 with spin up and particle 2 with spin down? We have

$$P(S_{1z} = \frac{\hbar}{2}, S_{2z} = -\frac{\hbar}{2}) = |\langle +-|\psi\rangle|^2 = \frac{1}{2}.$$

You can now easily infer that – in the singlet state – when particle 1 is determined to have spin up along z , then particle 2 will have spin down along z with certainty. We say that

* Well, it's "easy" if you realize that, in a state $|\psi\rangle$, the probability for getting an eigenvalue a of an operator A is (exercise)

$$P(a) = \sum_{i=1}^d |\langle i|\psi\rangle|^2,$$

where $|i\rangle$ are a basis for the d -dimensional subspace of vectors with eigenvalue a :

$$A|i\rangle = a|i\rangle, \quad i = 1, 2, \dots, d.$$

the S_z variables for particles 1 and 2 are “completely correlated”. Another way to state this is the following. If particle 1 is determined to have spin up along z , then we know two things with certainty - the z component of total spin is zero, and particle one has spin up. (Note that $[S_z, S_{1z}] = 0$.) The state vector of the system is given by $|+-\rangle$. A subsequent measurement of the z component of spin for particle 2 is then going to give spin-down with certainty. The foregoing discussion will work for any choice of axis, that is, it holds for any component of the spin (the same component being measured by each observer).

We imagine repeating this experiment many times. Each observer measures the single particle spin component and sees a random sequence of spin up or spin down results with a 50-50 probability. Quantum mechanics says that this is the best you can do – you cannot predict with certainty the S_z value for each particle because the individual particle spin is not compatible with the magnitude of the total spin of the system, which is what is known with certainty. The (singlet) state vector has been specified and there is no more information to be had about the system. On the other hand, if the two observers compare their data, they will see a perfect correlation between their data as described above. How do the particles “know” that they have to have a perfect correlation in their spin- z components, given that they can be separated by arbitrarily large distances? EPR say: each of the particles *does* have a definite value of the z component of spin when they are created, and these values are correlated as described above. It’s just that quantum mechanics is unable to predict the spin values for each particles in the first place and assigns them a 50-50 probability distribution. Thus EPR ascribe a certain “reality” to the spin component properties of each particle.

EPR go on to argue that if quantum mechanics is the ultimate description (no further information available) then a paradox arises. To see this, reason as follows. If both observers make measurements, then observers 1 and 2 each see a random distribution of spin up and spin down results, with a 50-50 probability. But when they compare their data they see that each observer 1 spin up is matched with an observer 2 spin down and vice versa. Thus, in this particular situation, one could say that particles 1 and 2 have in each experimental run a definite value of S_z . (Of course, the choice of the z axis is completely arbitrary.) Now, suppose that observer 1 chooses to measure S_x instead of S_z . When observer 1 gets spin up/down along x , then the state of the system is (exercise) $|S_{1x} = \pm\hbar/2; S_{2x} = \mp\hbar/2\rangle$ and observer 2’s S_z measurement will now yield spin up and spin down with a 50-50 probability. To see this, the probability $P(S_{2z} = \hbar/2)$ for getting $S_{2z} = \hbar/2$ in either state $|S_{1x} = \pm\hbar/2; S_{2x}, \mp\hbar/2\rangle$ can be computed by

$$\begin{aligned} P(S_{2z} = \hbar/2) &= |\langle S_{1x} = \hbar/2; S_{2z} = \hbar/2 | S_{1x} = \pm\hbar/2; S_{2x}, \mp\hbar/2 \rangle|^2 \\ &\quad + |\langle S_{1x} = -\hbar/2; S_{2z} = \hbar/2 | S_{1x} = \pm\hbar/2; S_{2x}, \mp\hbar/2 \rangle|^2 \\ &= \frac{1}{2}. \end{aligned}$$

Evidently, in this case, even knowing that, say, particle one has spin up along x , particle 2 does not have a definite value for S_z it has a 50-50 probability distribution. On the other hand, if observer 2 chose to measure S_x , with certainty the result will be opposite to the outcomes of observer 1's measurement. In other words, when observer 1 measures S_x , particle 2 can be viewed as having a definite value for S_x , but not of S_z . Thus it appears that the “reality” of particle 2 seems to depend upon the causally disconnected choices of Observer 1. A paradox.

Put differently we can say: (i) A measurement of one spin component yields random (50-50) values. (ii) When observer 1 measures S_z observer 2's S_z value is perfectly correlated with observer 1's measurement. (iii) When observer 1 measures S_x , observer 2's S_z values are random – uncorrelated with observer 1's results. Somehow particle 2 has to “know” whether to correlate or not with particle 1 — instantaneously!

To make this last paragraph more precise we can define the *correlation function* of two observables A and B in a given state to be the normalized covariance function:

$$C(A, B) = \left| \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\Delta A \Delta B} \right|.$$

The correlation function takes values in $[0, 1]$. An observable is always perfectly correlated with itself, $C(A, A) = 1$, while two statistically independent observables are uncorrelated: $C(A, B) = 0$. Using the singlet state we see that for any a

$$C(S_{1a}, S_{2a}) = 1,$$

indicating perfect correlation. On the other hand,

$$C(S_z, S_x) = 0,$$

Indicating uncorrelated observables.

From this discussion it is clear that – according to the description provided by quantum mechanics – one cannot assert that the spin observables were a part of the reality of each particle independent of the measurements. That's just how nature works (according to quantum mechanics). This is the resolution of the paradox, but we must accept the fact that quantum mechanics predicts a striking “non-locality”.

EPR argue that this non-locality is unacceptable in a complete description of the system and the only way out is to suppose that all the spin components of each particle are in fact well-defined, determined with certainty by the state preparation process, and quantum mechanics is just an incomplete theory with a concomitant bit of non-locality in its predictions.

Remarkably, it is possible to devise experiments (based upon *Bell inequalities*—see your text) which can test whether (1) observables do not have an *a priori* reality and nature is in

fact non-local in the sense provided by quantum mechanics or (2) nature is fundamentally local, observables have an *a priori* reality and hence a more complete (non-probabilistic) theory than quantum mechanics is possible. So far all experiments have confirmed the quantum mechanical description—nature really is as weird as quantum mechanics says it is!