Lecture 31

Relevant sections in text: §3.7

Example: A particle in 3-d with spin 1/2

We now apply these ideas and obtain some formulas describing the angular momentum of a particle moving in 3-d with spin 1/2. Of course this is a model of a non-relativistic electron so it is extremely important.

To begin, we specify the Hilbert space of states. Using the tensor product construction, we can view it as the space of linear combinations of the product basis:

$$|\vec{x}, \pm\rangle \equiv |\vec{x}\rangle \otimes |S_z, \pm\rangle.$$

A general state is then of the form

$$|\psi\rangle = \int d^3x \; \left(a_+(\vec{x})|\vec{x}, +\rangle + a_-(\vec{x})|\vec{x}, -\rangle\right).$$

As usual, $|a_\pm(\vec{x})|^2$ is the probability density for finding the particle at $\vec{x}$ with spin up/down along the z axis. Note that the normalization condition is

$$\int d^3x \; \left(|a_+|^2 + |a_-|^2\right) = 1.$$

As usual, we can characterize the state vector in terms of its components in the basis defined above. In this case we organize the information into a 2-component column vector whose entries are complex valued functions. This gadget is known as a *spinor field*:

$$\begin{pmatrix} \langle\vec{x}, +|\psi\rangle \\ \langle\vec{x}, -|\psi\rangle \end{pmatrix} = \begin{pmatrix} a_+(\vec{x}) \\ a_-(\vec{x}) \end{pmatrix}.$$

The position, momentum and spin operators are defined on the product basis as follows:

$$\vec{X}(|\vec{x}, \pm\rangle) = \vec{x}|\vec{x}, \pm\rangle, \quad \vec{P}|\vec{x}, \pm\rangle = (\vec{P}|\vec{x}\rangle) \otimes |S_z, \pm\rangle,$$

$$\vec{S}(|\vec{x}, \pm\rangle) = |\vec{x}\rangle \otimes (\vec{S}|S_z, \pm\rangle).$$

This implies that on the spinor fields the position and momentum operators do their usual thing ($\vec{X}$ multiplies, $\vec{P}$ differentiates) on each component function while the spin operators do their usual thing via 2×2 matrices. For example, the orbital angular momentum is $\vec{L} = \vec{X} \times \vec{P}$ and acts as

$$\vec{L}\begin{pmatrix} a_+(\vec{x}) \\ a_-(\vec{x}) \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{i}\vec{x} \times \nabla a_+(\vec{x}) \\ \frac{\hbar}{i}\vec{x} \times \nabla a_-(\vec{x}) \end{pmatrix},$$

while the spin acts via 2×2 matrices, *e.g.,*

$$S_x \begin{pmatrix} a_+(\vec{x}) \\ a_-(\vec{x}) \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_+(\vec{x}) \\ a_-(\vec{x}) \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a_-(\vec{x}) \\ a_+(\vec{x}) \end{pmatrix}.$$

States in which the orbital and spin angular momenta are each known with certainty are the product states

$$|l, s, m_l, m_s\rangle \equiv |l, m_l\rangle \otimes |m_s\rangle,$$

where $s = 1/2$ and $m_s = \pm\frac{1}{2}$. These states satisfy

$$L^2|l, s, m_l, m_s\rangle = l(l+1)\hbar^2|l, s, m_l, m_s\rangle, \quad L_z|l, s, m_l, m_s\rangle = m_l\hbar|l, s, m_l, m_s\rangle,$$

$$S_z|l, s, m_l, m_s\rangle = m_s\hbar|l, s, m_l, m_s\rangle.$$

We can now define the total angular momentum of the system as the operator

$$\vec{J} = \vec{L} + \vec{S}.$$

As usual, because $\vec{L}$ and $\vec{S}$ commute and satisfy the angular momentum commutation relations, we have

$$[J_a, J_b] = i\hbar\epsilon_{abc}J_c,$$

so the total angular momentum has all the general properties we deduced previously. For example, $\vec{J}$ generates rotations of the system as a whole, while $\vec{L}$ only generates rotations of position and momentum, and while $\vec{S}$ only generates rotations of the spin. We can only simultaneously diagonalize L^2, S^2, J^2 and one component, say, J_z. Setting $\vec{J}_1 = \vec{L}$ and $\vec{J}_2 = \vec{S}$ we have $j_1 = l = 0, 1, 2, \ldots$ and $j_2 = s = \frac{1}{2}$. For a state specified by $|l, s = 1/2, j, m\rangle$ we then have that the possible values for j are

$$j = l - \frac{1}{2}, l + \frac{1}{2}.$$