Lecture 30

Relevant sections in text: §3.7, 6.1, 6.2, 6.3

Some remarks on identical particles

This is a good place to point out that there is yet another postulate in quantum mechanics which deals with *identical particles.* These are particles that are intrincsically alike (same mass, spin, electric charge, *etc.*). Thus, for example, all electrons are identical. This does not mean that electrons cannot be distinguished *literally*, since we can clearly distinguish between an electron here on earth and one on the sun. But the point of view is that these are two (identical) electrons in different (position) states. The idea is that we view these particles as interchangeable in the sense that if, while you weren't looking, someone took the electron from the sun and swapped it with the one here on Earth (putting them in the respective states), then you couldn't tell the difference. This intrinsic indistinguishability of identical particles means the states of multi-particle systems must reflect this symmetry under particle interchange. This particle interchange can be represented as a unitary transformation which exchanges particles. *All probability distributions must be unchanged under this unitary transformation.* This means the state vector must change only by a phase when two identical particles are interchanged. It can be shown that it is sufficient to consider the possibility that this phase is ± 1. The new postulate of quantum mechanics (which can more or less be *derived* from relativistic quantum field theory) is that identical particles with integer spin ("bosons") should have state vectors which do not change under particle interchange (phase factor $+1$—"even" under exchange). For identical particles with half-integer spin ("fermions") the state vectors should change sign (phase factor -1—"odd under exchange"). This means that when considering systems of identical particles only a vector subspace of the tensor product Hilbert space is used to characterize states.* Moreover, this postulate implies the famous "Pauli exclusion principle" which asserts that no two electrons may occupy the same single particle state.

Let us see how this goes in our example of two spin 1/2 systems. Assuming our spin 1/2 particles are identical, and just taking account of the spin degrees of freedom (*i.e.,* ignoring translational degrees fo freedom), this means that none of the product states are allowed! This is because they are either even under particle interchange—the $|++\rangle$ and $|--\rangle$ states—or they are not invariant up to a phase—the $|+-\rangle$ and $|-+\rangle$ states. On the other hand, you can easily see that the *total* spin states of two spin 1/2 systems are invariant up to a phase. The triplet states are even under particle interchange; the singlet state is odd under particle interchange. If the two particles are identical and no other

* In the language of tensors, the Hilbert space of state vectors for identical bosons (fermions) is the symmetric (antisymmetric) tensor product.

degrees of freedom are present, then one must use the anti-symmetric singlet state only. This is the only possible state in this simple model! Note also that this state satisfies the exclusion principle.

Of course, real particles have translational degrees of freedom and the state vector will reflect that. As we shall discuss in detail soon, we can characterize a pair of electrons with translational degrees of freedom using a tensor product space with basis $|\vec{x}_1\rangle \otimes |\vec{x}_2\rangle \otimes |\pm\rangle \otimes |\pm\rangle$. States are superpositions of these product states. Only the *total* state vector must have the appropriate symmetry and this means one must consider both the position portion of the state as well as the spin portion. Thus it is possible to have a system of two electrons in a triplet state provided the position portion of the state vector is anti-symmetric under particle interchange.

Angular momentum addition in general

We can generalize our previous discussion of 2 spin 1/2 systems as follows. Suppose we are given two angular momenta $\vec{J}_1$ and $\vec{J}_2$ (*e.g.,* two spins, or a spin and an orbital angular momentum, or a pair of orbital angular momenta). We can discuss both angular momenta at once using the direct product space as before, with a product basis $|j_1, m_{1j}\rangle \otimes |j_2, m_{2j}\rangle$. We represent the operators on product vectors as

$$\vec{J}_1(|\alpha\rangle \otimes |\beta\rangle) = (\vec{J}|\alpha\rangle) \otimes |\beta\rangle,$$

and

$$\vec{J}_2(|\alpha\rangle \otimes |\beta\rangle) = |\alpha\rangle \otimes (\vec{J}|\beta\rangle),$$

and extend to general vectors by linearity. The product basis $|j_1, m_{1j}\rangle \otimes |j_2, m_{2j}\rangle$ is the basis corresponding to the commuting observables provided by $(J_1^2, J_2^2, J_{1z}, J_{2z}))$.

The total angular momentum is defined by

$$\vec{\mathbf{J}} = \vec{J}_1 + \vec{J}_2.$$

A set of commuting observables that includes the total angular momentum is provided by the operators $(J_1^2, J_2^2, \mathbf{J}^2, \mathbf{J}_z)$. Note that both bases are eigenvectors of J_1^2 and J_2^2 since these commute with all components of the individual and total angular momentum (exercise). We also note that product eigenvectors $|j_1, j_2, m_{1j}, m_{2j}\rangle$ are in fact eigenvectors of J_z with eigenvalues given by $m = m_{1j} + m_{2j}$ since

$$J_z|j_1, j_2, m_{1j}, m_{2j}\rangle = (J_{1z} + J_{2z})|j_1, j_2, m_{1j}, m_{2j}\rangle = (m_{1j} + m_{2j})\hbar|j_1, j_2, m_{1j}, m_{2j}\rangle.$$

But we will have to take linear combinations of product basis vectors to get total angular momentum vectors – eigenvectors of $\mathbf{J}^2$.

The basis of total angular momentum eigenvectors are denoted $|j_1, j_2, j, m_j\rangle$. For given values of j_1 and j_2, it can be shown (see the text) that the values for j are given by

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2$$

with (as usual)

$$m_j = -j, -j + 1, \ldots j - 1, j.$$

You can easily check that our results for the two spin 1/2 systems can be obtained from these formulas.

The two sets of commuting observables defining each kind of basis are not all compatible. In particular, J_{1z} and J_{2z} do not commute with J^2. So the set of total angular momentum eigenvectors, $|j_1, j_2, j, m_j\rangle$, will be distinct from the set of eigenvectors of the individual angular momenta, $|j_1, j_2, m_{1j}, m_{2j}\rangle$ (though some elements of each set may be the same). Total angular momentum eigenvectors, $|j_1, j_2, j, m_j\rangle$ can be expressed as linear combinations of $|j_1, j_2, m_1, m_2\rangle$ where the superposition will go over various m_{1j}, m_{2j} values. Of course one can also expand $|j_1, j_2, m_{1j}, m_{2j}\rangle$ in terms of $|j_1, j_2, j, m_j\rangle$ where the superposition will be over various j, m_j values. The coefficients in these superpositions are known as the *Clebsch-Gordan coefficients*. We have worked out a very simple example of all this in the case of a pair of spin 1/2 systems. There is a general theory of Clebsch-Gordan coefficients which we shall not have time to explore. Instead we will briefly visit another, relatively simple, and relatively important example.