Lecture 29

Relevant sections in text: §3.7

Total angular momentum (cont.)

To determine the linear combinations of the $m_s = 0$ product vectors $|+-\rangle$ and $|-+\rangle$ that yield $\mathbf{S}^2$ eigenvectors we use the angular momentum ladder operators:

$$\mathbf{S}_\pm = S_x \pm iS_y = S_{1\pm} + S_{2\pm}.$$

If we apply $\mathbf{S}_-$ to the eigenvector

$$|s = 1, m_s = 1\rangle = |++\rangle,$$

we get (exercise)

$$|s = 1, m_s = 0\rangle = \mathbf{S}_-|s = 1, m = 1\rangle = \mathbf{S}_-|++\rangle = (S_{1-}+S_{2-})|++\rangle = \frac{1}{\sqrt{2}}(|-+\rangle+|+-\rangle).$$

The other eigenket $|s = 0, m_s = 0\rangle$ must be orthogonal to this vector as well as to the other eigenkets, $|++\rangle$ and $|--\rangle$, from which its formula follows:

$$|s = 0, m_s = 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).$$

All together, we find the *total* angular momentum eigenvectors, $|s, m_s\rangle$, are related to the *individual* angular momentum (product) eigenkets by:

$$\begin{aligned}
|1, 1\rangle &= |++\rangle \\
|1, 0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \\
|1, -1\rangle &= |--\rangle, \\
|0, 0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).
\end{aligned}$$

These vectors form an orthonormal basis for the Hilbert space, so they are all the linearly independent eigenvectors of $\mathbf{S}^2$ and $\mathbf{S}_z$. The eigenstates with $s = 1$ are called the *triplet* states and the eigenstate with $s = 0$ is the *singlet* state. Notice that by combining two systems with half-angular momentum we end up with a system that allows integer angular momentum only.

A lengthier – but more straightforward – derivation of the eigenvectors $|s, m_s\rangle$ arises by simply writing the 4×4 matrix for $\mathbf{S}^2$ in the basis of product vectors $|\pm\pm\rangle$, and solving its eigenvalue problem. This is a good exercise. To get this matrix, you use the formula

$$\mathbf{S}^2 = \mathbf{S}_z^2 + +\hbar\mathbf{S}_z + \mathbf{S}_+\mathbf{S}_-.$$

It is straightforward to deduce the matrix elements of this expression among the product states since each of the operators has a simple action on those vectors. You should definitely try this computation if you want to get a better feel for what is going on.

Notice that the states of definite *total* angular momentum, $|s, m_s\rangle$, are not all the same as the states of definite individual angular momentum, say, $|\pm\pm\rangle$. This is because the total angular momentum is not compatible with the individual angular momenta. For example,

$$[\mathbf{S}^2, S_{1y}] = [S_1^2 + S_2^2 + 2(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}), S_{1y}] \neq 0.$$

This incompatibility has some striking physical consequences which we shall discuss a little later.

We have uncovered a couple of complete sets of commuting observables; they are given by $(S_1^2, S_2^2, S_{1z}, S_{2z}))$ and $(S_1^2, S_2^2, \mathbf{S}^2, \mathbf{S}_z)$. The eigenvectors of the first set are the product basis $|m_1, m_2\rangle = |\pm\pm\rangle$, representing states in which each individual spin angular momentum state is known with certainty. The eigenvectors of the second set are given by the singlet and triplet states, which are states in which the total angular momentum is known with certainty.