*Lecture 28*

*Relevant sections in text: §3.7*

**Two spin 1/2 systems: observables**

We have constructed the 4-d Hilbert space of states for a system consisting of two spin 1/2 particles. We built the space from the basis of "product states" corresponding to knowing the spin along $z$ for each particle with certainty. General states were, however, not necessarily products but rather superpositions of such. How are the observables to be represented as Hermitian operators on this space? To begin, let us consider the spin observables for each of the particles. Call them $\vec{S}_i = (\vec{S}_1, \vec{S}_2)$. We define them on product states via

$$\vec{S}_1(|\alpha\rangle \otimes |\beta\rangle) = (\vec{S}|\alpha\rangle) \otimes |\beta\rangle,$$

and

$$\vec{S}_2(|\alpha\rangle \otimes |\beta\rangle) = |\alpha\rangle \otimes (\vec{S}|\beta\rangle).$$

Here the operators $\vec{S}$ are the usual spin 1/2 operators (acting on a two-dimensional Hilbert space) that we have already discussed in some detail.

The action of $\vec{S}_1$ and $\vec{S}_2$ are defined on general vectors by expanding those vectors in a product basis, such as we considered earlier, and then using linearity to evaluate the operator term by term on each vector in the expansion. Sometimes one writes

$$\vec{S}_1 = \vec{S} \otimes I, \quad \vec{S}_2 = I \otimes \vec{S}$$

to summarize the above definition. The two spin operators $\vec{S}_1$ and $\vec{S}_2$ so-defined will commute (exercise) and have the same eigenvalues as their 1-particle counterparts (exercise). In detail, if $|\alpha\rangle$ is an eigenvector of some component of the spin, then so is $|\alpha\rangle \otimes |\beta\rangle$ for any $|\beta\rangle$. This means that if we know the spin component with certainty for particle one then we get an eigenvector of the corresponding component of $\vec{S}_1$, as we should. The same remarks apply to particle 2. Thus, as usual, when we expand a general vector in a product basis,

$$|\psi\rangle = a_{++}|S_z,+\rangle\otimes|S_z,+\rangle + a_{+-}|S_z,+\rangle\otimes|S_z,-\rangle + a_{-+}|S_z,-\rangle\otimes|S_z,+\rangle + a_{--}|S_z,-\rangle\otimes|S_z,-\rangle.$$

we have that $|a_{++}|^2$ is the probability for finding particle 1 and particle two to have spin up along $z$. Likewise we have $|a_{+-}|^2$ giving the probability for finding particle 1 to have spin up along $z$ and for particle 2 to have spin down along $z$. In this way we recover the usual properties of each particle, now viewed as subsystems.

Let us explore a simple example of some of this. Consider the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}|S_z,+\rangle \otimes |S_z,+\rangle - \frac{1}{\sqrt{2}}|S_z,+\rangle \otimes |S_z,-\rangle.$$

Let us work out some probabilities. To begin with the probability for finding spin up or down along $z$ for particle one is 1 or 0 respectively. The probability for getting spin up or down along $z$ for particle 2 is 1/2 for either possibility. You can read these probabilities off of the formula for $|\psi\rangle$, or you can simply take the scalar product of $|\psi\rangle$ with the relevant eigenvectors (taking due account of degeneracy!!) as usual. Let us try something a little harder. What is the probability that particle one has spin up along $z$ and particle 2 has spin down along $x$? There are a couple of ways to think about computing the answer to this question: (1) you can expand $|\psi\rangle$ in the product basis of $S_z$ and $S_x$ eigenvectors for particles one and two respectively, (2) you can take the scalar product of $|\psi\rangle$ with the appropriate eigenvectors. Let's use the second method. You can easily check that $|S_z,+\rangle\otimes|S_x,-\rangle$ is the normalized eigenvector of $S_{1z}$ and $S_{2x}$ with the desired eigenvalues. Thus the probability is

$$|\langle S_z,+|\otimes S_x,-|\psi\rangle|^2 = |\frac{1}{\sqrt{2}}\langle S_z,+|S_z,+\rangle\langle S_x,+|S_z,-\rangle - \frac{1}{\sqrt{2}}\langle S_z,+|S_z,+\rangle\langle S_x,-|S_z,-\rangle|^2 = 1.$$

So, $|\psi\rangle$ represents a state where $S_{1z}$ and $S_{2x}$ are known with certainty! As you know, this means that this vector must be a simultaneous eigenvector of these two (commuting) operators. This is indeed the case. You can easily check that in this example $|\psi\rangle$ happens to be the product vector:

$$|\psi\rangle = |S_z,+\rangle\otimes|S_x,-\rangle.$$

**Total angular momentum**

We have shown how to define observables which refer to the individual subsystems (spins in our example). There are other observables that are only defined for the composite system. Consider the *total* angular momentum $\vec{\mathbf{S}}$, defined by

$$\vec{\mathbf{S}} = \vec{S}_1 + \vec{S}_2.$$

You can easily check that this operator is Hermitian and that

$$[\mathbf{S}_k, \mathbf{S}_l] = i\hbar\epsilon_{klm}\mathbf{S}_m,$$

so it does represent angular momentum. Indeed, this operator generates rotations of the two particle system as a whole. The individual spin operators $\vec{S}_1$ and $\vec{S}_2$ only generate rotations of their respective subsystems.

Using our general theory of angular momentum we know that we can find a basis of common eigenvectors of $\mathbf{S}^2$ and any one component, say, $\mathbf{S}_z$. Let us write these as $|s, m_s\rangle$, where

$$\mathbf{S}^2|s,m_s\rangle = s(s+1)\hbar^2|s,m_s\rangle, \quad \mathbf{S}_z|s,m_s\rangle = m_s\hbar|s,m_s\rangle.$$

Of course, *a priori* we know that $s$ and $m_s$ can be integers or half-integers. We shall see that only certain integer values actually occur. Let us define the basis

$$|\pm,\pm\rangle = |S_z,\pm\rangle \otimes |S_z,\pm\rangle.$$

This product basis physically corresponds to states in which the $z$ component of spin for each particle is known with certainty. In the following we will find the total angular momentum eigenvalues and express the eigenvectors in terms of the product basis $|\pm,\pm\rangle$.

To begin with, it is clear that the product basis above is in fact the basis of eigenvectors of $\mathbf{S}_z$. To see this, we set $m_1 = \pm\frac{1}{2}$, $m_2 = \pm\frac{1}{2}$, so that the product basis vectors $|\pm,\pm\rangle$ can be denoted by $|m_1, m_2\rangle$, and we compute

$$\mathbf{S}_z|m_1, m_2\rangle = (S_{1z} + S_{2z})|m_1, m_2\rangle = (m_1 + m_2)\hbar|m_1, m_2\rangle.$$

Evidently, $m_s = -1, 0, 1$ with $m = 0$ being doubly degenerate (exercise). From our general results on angular momentum it is clear that the only possible values for the total spin quantum number are therefore $s = 0, 1$. From this we can infer that the $m_s = \pm 1$ eigenvectors must be $\mathbf{S}^2$ eigenvectors with $s = 1$, but we may need linear combinations of the $m_s = 0$ product eigenvectors to get $\mathbf{S}^2$ eigenvectors. To see why the vectors $|++\rangle$ and $|--\rangle$ must be also $\mathbf{S}^2$ eigenvectors one reasons as follows. Our general theory guarantees us the existence of a basis of simultaneous $\mathbf{S}^2$ and $\mathbf{S}_z$ eigenvectors. It is easy to see that the $|++\rangle$ and $|--\rangle$ are the only eigenvectors (up to normalization) with $m_s = \pm 1$, since any other vectors can be expanded in the product basis and this immediately rules out any other linear combinations (exercise). Therefore, these two vectors must be the $\mathbf{S}^2$ eigenvectors. Because they have $m_s = \pm 1$ and we know that $s = 0, 1$ it follows that the $|++\rangle$ and $|--\rangle$ vectors are $\mathbf{S}^2$ eigenvectors with $s = 1$.

*To be continued...*