Lecture 27

Relevant sections in text: §3.6, 3.7

Orbital angular momentum eigenvalues and eigenfunctions; spherical harmonics

We are solving the equations

$$L_z\psi(r,\theta,\phi) \equiv \frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi(r,\theta,\phi) = m_l\psi(r,\theta,\phi)$$

$$L^2\psi(r,\theta,\phi) \equiv -\hbar^2\left(\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\right)\psi(r,\theta,\phi) = l(l+1)\hbar^2\psi(r,\theta,\phi).$$

We have already seen the solution to the L_z equation takes the form

$$\psi(r,\theta,\phi) = f_{l,m_l}(r,\theta)e^{im_l\phi},$$

where m_l—and hence l— must be an integer.

Having solved the L_z equation we now must solve the L^2 equation, which is an ordinary differential equation for $f_{lm_l}(r,\theta)$:

$$\frac{1}{\hbar^2}L^2 f_{lm_l} = \left(\frac{m_l^2}{\sin^2\theta} - \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\right)f_{l,m_l} = l(l+1)f_{lm_l}.$$

The solutions of this equation are the associated Legendre polynomials $P_{l,m_l}(\cos\theta)$ and the angular momentum eigenfunctions are thus of the form

$$\psi_{l,m_l}(r,\theta,\phi) = \tilde{f}_{l,m_l}(r)Y_{l,m_l}(\theta,\phi),$$

where the $Y_{l,m}(\theta,\phi)$ are the *spherical harmonics*,

$$Y_{l,m}(\theta,\phi) = N_{l,m_l}P_{l,m_l}(\cos\theta)e^{im_l\phi}.$$

Here N_{l,m_l} is a normalization constant (see below) and the functions $\tilde{f}_{lm_l}(r)$ are the "integration constants" for the solution to the purely angular differential equations. See your text for detailed formulas for the spherical harmonics. Note that all non-negative integer values are allowed for l. As discussed earlier, the functions $\tilde{f}_{l,m_l}(r)$ are not determined by the angular momentum eigenvalue problem. Typically these functions are fixed by requiring the wave function to be also an eigenfunction of another observable which commutes with L^2 and L_z, *e.g.,* the energy in a central force problem. In any case, we will assume that

$$\int_0^\infty dr\, r^2|f_{l,m_l}(r)|^2 = 1.$$

This way, with the conventional normalization of the spherical harmonics:

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \, \sin^2\theta Y^*_{l',m_l'}(\theta,\phi) Y_{l,m_l}(\theta,\phi) = \delta_{ll'}\delta_{m_l m_l'},$$

we have that

$$\langle l', m_l' | l, m_l\rangle = \int_0^\infty dr\, r^2 \int_0^\pi d\theta \int_0^{2\pi} d\phi, \sin^2\theta \psi^*_{l',m_l'}(r,\theta,\phi)\psi_{l,m_l}(r,\theta,\phi) = \delta_{ll'}\delta_{m_l m_l'}.$$

For a state of definite angular momentum $|l, m_l\rangle$ we see that the angular dependence of the probability distribution is completely determined by the spherical harmonics. The radial dependence of the probability distribution is not determined by the value of angular momentum unless other requirements are made upon the state.

Addition of angular momentum: Two spin 1/2 systems

We now will have a look at a rather important and intricate part of angular momentum theory involving the combination of two (or more) angular momenta. We will first focus on the problem of making a quantum model for a system consisting of two distinguishable spin 1/2 particles (ignoring all but their spin degrees of freedom). The idea is simply to combine two copies of our existing model of a spin 1/2 system. The technology we shall need is the *tensor product* construction.* We shall introduce this construction in the context of the problem of combining – or "adding" – two spin 1/2 angular momenta.

For a system of two spin 1/2 particles, *e.g.,* an electron and a positron, we can imagine measuring the component of spin for each particle along a given axis, say the z axis. (We can use different axes for each particle if we like.) Obviously there are 4 possible outcomes (exercise); we can denote the states in which these spin values are known with certainty by

$$|S_z,+\rangle\otimes|S_z,+\rangle, \quad |S_z,+\rangle\otimes|S_z,-\rangle, \quad |S_z,-\rangle\otimes|S_z,+\rangle, \quad |S_z,-\rangle\otimes|S_z,-\rangle.$$

Here the first factor of the pair always refers to "particle 1" and the second factor refers to "particle 2". We view these vectors as an orthonormal basis for a new Hilbert space of states describing the two particle system; this is the tensor product Hilbert space. We thus consider the 4-d Hilbert space of formal linear combinations of these 4 basis vectors. An arbitrary vector $|\psi\rangle$ is given by

$$|\psi\rangle = a_{++}|S_z,+\rangle\otimes|S_z,+\rangle + a_{+-}|S_z,+\rangle\otimes|S_z,-\rangle + a_{-+}|S_z,-\rangle\otimes|S_z,+\rangle + a_{--}|S_z,-\rangle\otimes|S_z,-\rangle.$$

Here the scalar multiplication is assigned to the pair as a whole, but by definition it can be assigned to either of the factors in the pair as well. If you wish you can view the scalars

* The text misleadingly calls this construction the "direct product".

$a_{\pm\pm}$ as forming a column vector with 4 rows; the squares of these scalars give the various probabilities for the outcome of the S_z measurement for each particle. Other bases are possible, corresponding to other experimental arrangements, *e.g.,* S_x for particle 1 and S_y for particle 2.

In the above discussion we introduced a special class of states: the *product states*, which can be expressed as a pair $|\psi\rangle \otimes |\phi\rangle$. Each element of the basis shown above is a product state; a general state vector can be expressed as a superposition of product states. We define

$$(|\alpha\rangle + |\beta\rangle) \otimes |\gamma\rangle = |\alpha\rangle \otimes |\gamma\rangle + |\beta\rangle \otimes |\gamma\rangle,$$

and

$$|\gamma\rangle \otimes (|\alpha\rangle + |\beta\rangle) = |\gamma\rangle \otimes |\alpha\rangle + |\gamma\rangle \otimes |\beta\rangle.$$

The dual space to the tensor product space can be defined as follows. The action of a product bra $\langle\alpha| \otimes \langle\beta|$ on a product ket $|\gamma\rangle \otimes |\delta\rangle$ is given by

$$(\langle\alpha| \otimes \langle\beta|)(|\gamma\rangle \otimes |\delta\rangle) := \langle\alpha|\gamma\rangle\langle\beta|\delta\rangle,$$

and the action of a general bra on a general ket is defined by expanding each in terms of the product basis, expanding out the result using linearity, and then using the above definition term by term. The rules for scalar multiplication, addition, *etc.* for bras is identical to that for kets.

The scalar product on the Hilbert space is such that

$$(|\alpha\rangle \otimes |\beta\rangle)^\dagger = \langle\alpha| \otimes \langle\beta|.$$

This implies that the basis shown above is orthonormal (nice exercise!).

In general we can play a similar game with any 2 vector spaces, say, V of dimension m and a vector space W of dimension n, to get tensor product space, denoted by $V \otimes W$, with dimension mn.