Lecture 26

Relevant sections in text: §3.6

Position representation of angular momentum operators

We have seen that the position operators act on position wave functions $\psi(\vec{x})$ by multiplication and the momentum operators act by differentiation. It is not hard to see that, at any given point (except the origin), the angular momentum operators take derivatives only in directions orthogonal to the position vector for that point. We have

$$\vec{X} \cdot \vec{L}\psi = \vec{X} \cdot (\vec{X} \times \vec{P})\psi = 0.$$

Thus it is convenient to use spherical polar coordinates (r, θ, ϕ), when working with the orbital anguler momentum. We have (good exercise!)

$$L_x\psi(r, \theta, \phi) = \frac{\hbar}{i}\left(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi\right)\psi(r, \theta, \phi),$$

$$L_y\psi(r, \theta, \phi) = \frac{\hbar}{i}\left(-\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\right)\psi(r, \theta, \phi)$$

$$L_z\psi(r, \theta, \phi) = \frac{\hbar}{i}\partial_\phi\psi(r, \theta, \phi).$$

You can see that L_z is particularly simple – it clearly generates "translations" in ϕ, which are rotations about the z axis, of course. The other two components of $\vec{L}$ also generate rotations about their respective axes. They do not take such a simple form because spherical polar coordinates give the z axis special treatment.

Combining these results we have, in addition,

$$L^2\psi(r, \theta, \phi) = -\hbar^2\left(\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\right)\psi(r, \theta, \phi).$$

You may recognize that this last result is, up to a factor of $-\hbar^2 r^2$, the angular part of the Laplacian. This result arises from the identity (see text)

$$L^2 = r^2P^2 - (\vec{X} \cdot \vec{P})^2 + i\hbar\vec{X} \cdot \vec{P},$$

where $r^2 = X^2 + Y^2 + Z^2$, so that (exercise)

$$P^2\psi(r, \theta, \phi) = -\hbar^2\nabla^2\psi(r, \theta, \phi) = -\hbar^2(\frac{1}{\hbar^2r^2}L^2 + \partial_r^2 + \frac{2}{r}\partial_r)\psi(r, \theta, \phi).$$

Thus we get, in operator form, the familiar decomposition of kinetic energy into a radial part and an angular part.

Orbital angular momentum eigenvalues and eigenfunctions; spherical harmonics

A good way to see what is the physical content of the orbital angular momentum eigenvectors is to study the position probability distributions in these states. Thus we consider the position wave functions

$$\psi_{lm} = \langle \vec{x}|l, m\rangle$$

corresponding to orbital angular momentum eigenvectors. These are simultaneous eigenfunctions of L^2 and L_z, so they satisfy

$$L_z\psi_{lm_l} = m_l\hbar\psi_{lm_l}, \quad L^2\psi_{lm_l} = l(l+1)\hbar^2\psi_{lm_l},$$

where – on general grounds –

$$l = 0, \frac{1}{2}, 1, \ldots,$$

and

$$m_l = -l, -l+1, \ldots l-1, l.$$

We note that the angular momentum eigenfunctions will always involve an arbitrary multiplicative function of the radius r. This is because the angular momentum differential operators only take derivatives in the angular directions. What this means physically is that the states of definite angular momentum will always have a degeneracy. This should not surprise you: just specifying the angular momentum of a state of a particle is not expected to completely determine the particle's state.*

We now argue that the half-integer possibility does not occur for orbital angular momentum. First, we note that it is quite easy to find L_z eigenfunctions in spherical polar coordinates since $L_z = -i\hbar\partial_\phi$. Evidently,

$$\psi_{lm_l} = f_{lm_l}(r, \theta)e^{im_l\phi}.$$

Immediately we see that m_l can only be an integer – otherwise ψ_{lm} will not be a continuous function. Now, discontinuous wave functions are not inherently evil. Indeed, there are plenty of discontinuous functions in the Hilbert space. But these functions will fail to be differentiable and hence will not be in the domain of the momentum and angular momentum operators. We are trying to construct the angular momentum eigenfunctions which, by definition, are in the domain of the operators. Thus we can conclude immediately that, for orbital angular momentum, we can only have (at most)

$$l = 0, 1, 2, \ldots, \quad \text{and} \quad m_l = -l, -l+1, \ldots l-1, l.$$

As we shall see, all of the indicated values do in fact arise. (By the way, in your homework assignment you will find a problem which gives an alternative argument for this result.)

* For example, consider the classical motion of a particle with vanishing angular momentum. The motion is such that position and momentum vectors are parallel, but are otherwise arbitrary.