Lecture 25

Relevant sections in text: §3.5, 3.6

Spin systems in general

Let us note that the spin 1/2 observables, being angular momentum operators, must have eigenvectors/eigenvalues obeying the general results we have just derived. Indeed, you can easily see that with $j = 1/2$ we reproduce the standard results on the spectrum of the spin operators. For example we have

$$J^2 \leftrightarrow S^2 = \frac{3}{4}\hbar^2 I,$$

which has eigenvalues

$$\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2.$$

Given that $j = \frac{1}{2}$ we have

$$m_j = -\frac{1}{2}, \frac{1}{2},$$

so that the eigenvalues for $J_z \leftrightarrow S_z$ are $\pm\frac{\hbar}{2}$, as they should be.

The "1/2" in "spin 1/2" comes from the fact that $j = 1/2$ for all the states of this system. We can generalize this to other values of j. We speak of a particle or system having spin s if it admits angular momentum operators which act on a Hilbert space of states all of which have the same eigenvalue for J^2, that is, all of which have the same value $j = s$. For a system with spin-s and no other degrees of freedom the Hilbert space of states has dimension $2s + 1$ and the operator representing the squared-magnitude of the spin is given by (exercise)

$$J^2 = s(s+1)\hbar^2 I.$$

Usually in this situation we use the notation $\vec{S}$ for $\vec{J}$.

Orbital angular momentum

In nature it appears that angular momentum comes in two types when we use a "particle" description of matter. First there is the intrinsic "spin" angular momentum carried by an elementary particle. The spin (j) of a particle is fixed once and for all—although the spin state (m_j) is not—and is part of the essential attribute that makes a particle what it is.* Second, there is the "orbital" angular momentum which arises due to the motion of the particle in space. Both of these types of angular momentum are to some extent unified

* What I mean by this is, *e.g.*, an electron is characterized by its mass, its electric charge, its spin ($j = 1/2$) – and a few other things besides.

when using the presumably more fundamental description of matter and its interactions afforded by quantum field theory. Of course, we will be sticking to the notion of particle which arises in the framework of (non-relativistic) quantum mechanics.

The orbital angular momentum of a particle is represented by the operator

$$\vec{L} = \vec{X} \times \vec{P},$$

where $\vec{X}$ is the position operator relative to some fixed origin. Let us note that the ordering of the operators $\vec{X}$ and $\vec{P}$ is unambiguous since the cross product only brings commuting operators into play. For example,

$$L_z = XP_y - YP_x.$$

The formula above for $\vec{L}$ is familiar from classical mechanics, but it can be justified using the angular momentum commutation relations. For example, you can check by a straightforward computation that

$$[L_x, L_y] = [YP_z - ZP_y, ZP_x - XP_z] = i\hbar(XP_y - YP_x) = i\hbar L_z.$$

The other angular momentum commutation relations follow in a similar fashion.

Orbital angular momentum and rotations

To further justify this form of the orbital angular momentum, we can study its role as infinitesimal generator of rotations. Let us consider an infinitesimal rotation about the z-axis. The putative generator is

$$L_z = XP_y - YP_x.$$

We can study the action of L_z on states by computing its action on the positions basis,

$$|\vec{x}\rangle = |x, y, z\rangle, \quad \vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle.$$

An infinitesimal rotation by an angle ϵ is given by

$$D(\epsilon) = I - \frac{i}{\hbar}\epsilon L_z + \mathcal{O}(\epsilon^2),$$

so that, to first order in ϵ,

$$\begin{aligned} D(\epsilon)|x, y, z\rangle &= [I - \frac{i}{\hbar}\epsilon(xP_y - yP_x)]|x, y, z\rangle + \mathcal{O}(\epsilon^2) \\ &= |x - \epsilon y, y + \epsilon x, z\rangle + \mathcal{O}(\epsilon^2). \end{aligned}$$

Here we used the fact that momentum generates translations. Now recall the following geometric fact: under an infinitesimal rotation about an axis $\hat{n}$ by an angle ϵ the position vector (indeed, any vector) transforms as

$$\vec{x} \to \vec{x} + \epsilon \hat{n} \times \vec{x} + \mathcal{O}(\epsilon^2).$$

Choosing $\hat{n}$ along the z axis, we can compare this formula with the change of the position eigenvector under the infinitesimal transformation generated by L_z. We see that the position eigenvector's eigenvalue rotates properly (at least infinitesimally).

It is not hard to see that under a finite (*i.e.,* non-infinitesimal) rotation about the z-axis we have

$$\begin{aligned} e^{-\frac{i}{\hbar}\theta L_z}|x, y, z\rangle &= |x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta, z\rangle \\ &= |R(\hat{k}, \theta\rangle \cdot \vec{x}\rangle. \end{aligned}$$

You can prove this by iterating the infinitesimal transformation, for example. Since the z axis is arbitrary, we have in fact proved (exercise)

$$e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle = |R(\hat{n}, \theta\rangle \cdot \vec{x}\rangle.$$

This implies that (exercise)

$$e^{\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}\vec{X}e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}} = R(\hat{n}, \theta) \cdot \vec{X},$$

which can be checked by evaluating it on the position basis. Therefore we have that (exercise)

$$X(e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle) = R(\hat{n}, \theta) \cdot \vec{x}(e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle),$$

so that the rotation operator on the Hilbert space maps eigenvectors of position to eigenvectors with the rotated position:

$$e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle = |R(\hat{n}, \theta) \cdot \vec{x}\rangle.$$

From this result we have the position wave functions rotating properly (exercise):

$$e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}\psi(\vec{x}) = \langle\vec{x}|e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\psi\rangle = \psi(R^{-1}(\hat{n}, \theta) \cdot \vec{x}).$$

An identical set of results can be obtained for the momentum operators and their eigenvectors and momentum wave functions. This is a satisfactory set of results since the momentum *vector* should behave in the same way as the position vector under rotations. We have, in particular

$$D(\hat{n}, \theta)|\vec{p}\rangle = e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{p}\rangle = |R(\hat{n}, \theta) \cdot \vec{p}\rangle,$$

$$e^{\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}\vec{P}e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}} = R(\hat{n}, \theta) \cdot \vec{P}.$$