Lecture 24

Relevant sections in text: §3.5

Angular momentum in general

We can deduce quite a lot about the angular momentum $\vec{J}$ of any system just knowing that it is the generator of rotations, *i.e.,* represented by 3 self-adjoint operators satisfying the angular momentum commutation relations. To begin with, it is clear that the 3 components of $\vec{J}$ are not compatible so that, generally speaking, one will not be able to determine more than one component with certainty. Indeed, the only state in which 2 or more components of $\vec{J}$ are known with certainty is an eigenvector of all components with eigenvalue zero, *i.e.,* a state with vanishing angular momentum. To see this, suppose that $|\alpha\rangle$ is an eigenvector of J_x and J_y, then it is easy to see from

$$[J_x, J_y]|\alpha\rangle = i\hbar J_z|\alpha\rangle,$$

that $|\alpha\rangle$ is an eigenvector of J_z with eigenvalue 0. Now consider

$$[J_z, J_x]|\alpha\rangle = i\hbar J_y|\alpha\rangle.$$

You can easily see that the left hand side vanishes hence $|\alpha\rangle$ is an eigenvector of J_y with eigenvalue zero. Similarly you can show (exercise) that $J_x|\alpha\rangle = 0$ also. Thus, if there is any angular momentum in the system at all, at most one component can be known with certainty in any state. When we consider states with a definite value for a component of $\vec{J}$, we usually call that component J_z, by convention. But it is important to realize that there is nothing special about the z-direction; one can find eigenvectors for any one component of $\vec{J}$ (*cf.* spin 1/2).

We next observe that the (squared) magnitude of the angular momentum,

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

is a Hermitian operator (we will assume that J^2 is self-adjoint) that is compatible with any component J_i. To see this is a very simple computation:

$$[J^2, J_k] = \sum_l \left(J_l[J_l, J_k] + [J_l, J_k]J_l\right) = i\hbar \sum_{l,m} \epsilon_{lkm}(J_l J_m + J_m J_l) = 0,$$

where the last equality follows from the antisymmetry of ϵ_{lkm}.* Consequently, there exists an orthonormal basis of simultaneous eigenvectors of J^2 and any one component of $\vec{J}$

* The quantity in parenthesis is symmetric under $l \leftrightarrow m$ while ϵ_{lkm} is anti-symmetric when this interchange is performed. This guarantees that each term in the double sum will be canceled by another term in the double sum.

(usually denoted J_z). Physically, this means that while the 3 components of angular momentum are not compatible, there exists a complete set of states in which the magnitude of angular momentum and one component of angular momentum are known with certainty. As a simple example, consider the spin 1/2 case, where

$$S^2 = \frac{3}{4}\hbar^2 I,$$

which obviously commutes with each of the spin operators. Clearly any basis of eigenvectors of any of the S_i will also be eigenvectors of S^2.

Angular momentum eigenvalues and eigenvectors

Of course, given an observable represented as an operator, the most pressing business is to understand the spectral properties of the operator since its spectrum determines the possible outcomes of a measurement of the observable and the (generalized) eigenvectors are used to compute the probability distribution of the observable in a given state. In our case we have defined angular momentum as operators satisfying

$$\vec{J} = \vec{J}^\dagger, \quad [J_l, J_m] = i\hbar\epsilon_{lmn}J_n.$$

Just from these relations alone there is a lot we can learn about the spectral properties of angular momentum.

We assume that each of the operators J_i and J^2 admit eigenvectors. Let us study the angular momentum eigenvalues and eigenvectors, the latter being simultaneous eigenvectors of J_z and J^2. We write

$$J^2|a, b\rangle = a|a, b\rangle, \quad J_z|a, b\rangle = b|a, b\rangle,$$

The possible values of a and b can be deduced much in the same way as the spectrum of the Hamiltonian for an oscillator can be deduced using the raising and lowering operators. To this end we define the angular momentum *ladder operators*

$$J_\pm = J_x \pm iJ_y, \quad J_\pm^\dagger = J_\mp.$$

Of course, these two operators contain the same physical information as J_x and J_y. In terms of the ladder operators, the angular momentum commutation relations can be expressed as (exercise)

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [J_\pm, J^2] = 0, \quad [J_\pm, J_\mp] = \pm 2\hbar J_z.$$

From these relations we can see that the vector $J_\pm|a, b\rangle$ satisfies (exercise)

$$J^2(J_\pm|a, b\rangle) = a(J_\pm|a, b\rangle), \quad J_z(J_\pm|a, b\rangle) = (b \pm \hbar)(J_\pm|a, b\rangle).$$

Thus, when acting on angular momentum eigenvectors (eigenvectors of J^2 and J_z), the ladder operators preserve the magnitude of the angular momentum but increase/decrease the z component by a "quantum of angular momentum" $\hbar$.

Next we show that the eigenvalues of J^2 are non-negative and bound the magnitude of the eigenvalues of J_z. One way to see this arises by studying the relation

$$J^2 - J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+) = \frac{1}{2}(J_-^\dagger J_- + J_+^\dagger J_+).$$

Now, for any operator A and vector $|\psi\rangle$ we have that (exercise)

$$\langle\psi|A^\dagger A|\psi\rangle \geq 0,$$

so that for any vector $|\psi\rangle$ (in the domain of the squared angular momentum operators) (exercise)

$$\langle\psi|J^2 - J_z^2|\psi\rangle \geq 0.$$

Assuming the eigenvectors $|a,b\rangle$ are not of the "generalized" type, *i.e.,* are normalizable, we have

$$0 \leq \langle a,b|J^2 - J_z^2|a,b\rangle = a - b^2,$$

and hence

$$a \geq 0, \quad -\sqrt{a} \leq b \leq \sqrt{a}.$$

The ladder operators increase/decrease the b value of the eigenvector with out changing a. Thus by repeated application of these operators we can violate the inequality above unless there is a maximum and minimum value for b such that application of J_+ and J_-, respectively, will result in the zero vector. Moreover, if we start with an eigenvector with a minimum (maximum) value for b, then by successively applying J_+ (J_-) we must hit the maximum (minimum) value. As shown in your text, these requirements lead to the following results. The eigenvalues a can only be of the form

$$a = j(j+1)\hbar^2,$$

where $j \geq 0$ can be a non-negative integer or a half integer only:

$$j = 0, 1/2, 1, 3/2, \ldots.$$

For an eigenvector with a given value of j, the eigenvalues b are given by

$$b = m_j\hbar,$$

where

$$m_j = -j, -j+1, \ldots, j-1, j.$$

Note that if j is an integer then so is m_j, and if j is a half-integer, then so is m_j. Note also that for a fixed value of j there are $2j+1$ possible values for m_j. The usual notational convention is to denote angular momentum eigenvectors by $|j, m_j\rangle$, with j and m_j obeying the restrictions described above.

The preceding arguments show how the self-adjointness and commutation relations of angular momentum give plenty of information about the spectra. We note that these are necessary conditions, *e.g.,* the magnitude of angular momentum must be determined via an integer or half-integer, but this does not mean that all these possibilities will occur. As we shall see, for orbital angular momentum only the integer possibility is utilized. For the spin 1/2 system, a single value $j = 1/2$ is utilized. We will discuss this in a little more detail next.