Lecture 23

Relevant sections in text: §3.2

Spin as angular momentum

You will recall that the spin 1/2 observables S_i have the dimensions of angular momentum and satisfy the angular momentum commutation relations:

$$[S_k, S_l] = i\hbar\epsilon_{klm}S_m.$$

From our previous discussion we then know that we can interpret $\vec{S}$ as generators of rotations—angular momentum—and that a rotation of the system by an angle θ about an axis $\hat{n}$ corresponds to a change in the state of the spin 1/2 system accomplished by a unitary operator of the form

$$D(\hat{n}, \theta) = e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{S}}.$$

Let us have a look at an example.

Consider a rotation about the z-axis. Using the S_z eigenvectors as a basis we have the matrix elements

$$S_z \longleftrightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that the matrix elements of the rotation operator are (exercise)

$$D(\hat{z}, \theta) \longleftrightarrow \begin{pmatrix} e^{-\frac{i}{2}\theta} & 0 \\ 0 & e^{\frac{i}{2}\theta} \end{pmatrix}.$$

You can do this calculation by using the power series definition of the exponential – you will quickly see the pattern. You can also use the spectral decomposition definition. Recall that for any self-adjoint operator A we have

$$A = \sum_i a_i |i\rangle\langle i|,$$

where $A|i\rangle = a_i|i\rangle$. Similarly, as you saw in the homework,

$$f(A) = \sum_i f(a_i)|i\rangle\langle i|.$$

Let

$$A = S_z = \frac{\hbar}{2}(|+\rangle\langle +| - |-\rangle\langle -|),$$

and let f be the appropriate exponential function and you will get

$$D(\hat{z}, \theta) = e^{-\frac{i}{\hbar}\theta S_z} = e^{-\frac{i}{2}\theta}|+\rangle\langle +| + e^{\frac{i}{2}\theta}|-\rangle\langle -|.$$

In the $|\pm\rangle$ basis the matrix elements of D are as advertised.

Notice that this family of unitary operators satisfies

$$D(\hat{z}, \theta_1)D(\hat{z}, \theta_2) = D(\hat{z}, \theta_1 + \theta_2),$$

as it should. On the other hand, right away you should notice that something interesting has happened. The unitary transformation of a spin 1/2 system corresponding to a rotation by 2π is not the identity, but rather minus the identity! Thus, if you rotate a spin 1/2 system by 2π its state vector $|\psi\rangle$ transforms to

$$|\psi\rangle \to e^{-\frac{i}{\hbar}(2\pi)S_z}|\psi\rangle = -|\psi\rangle.$$

Indeed, it is only after a rotation by 4π that the spin 1/2 state vector returns to its original value. This looks bad; how can such a transformation rule agree with experiment? Actually, everything works out fine since the expectation values are insensitive to this change in sign:

$$\langle\psi|A|\psi\rangle \to \langle\psi|e^{\frac{i}{\hbar}(2\pi)S_z}Ae^{-\frac{i}{\hbar}(2\pi)S_z}|\psi\rangle = \langle\psi|(-1)A(-1)|\psi\rangle = \langle\psi|A|\psi\rangle.$$

What is happening here is that the spin 1/2 system is taking advantage of the phase freedom in the projective representation. On the one hand we have

$$R(\hat{n}, \theta)R(\hat{n}, 2\pi - \theta) = R(\hat{n}, 2\pi) = R(\hat{n}, 0) = I.$$

For the spin 1/2 representation we have

$$D(\hat{n}, \theta)D(\hat{n}, 2\pi - \theta) = -I = -D(\hat{n}, 0).$$

Let us note that the phase freedom in the representation of rotations is a rather subtle, intricate feature of the way in which quantum mechanics describes the physical world. Because of this subtlety in the theory we are able to properly accommodate spin 1/2 systems. This is one of the great successes of quantum mechanics.

How do observable quantities change in general when we make a rotation about z? Under a rotation about z the expectation value transforms via

$$\langle A\rangle = \langle\psi|A|\psi\rangle \to \langle\psi|e^{\frac{i}{\hbar}\theta S_z}Ae^{-\frac{i}{\hbar}\theta S_z}|\psi\rangle$$

Choose, for example, $A = S_x$. By expanding the exponentials in a Taylor series or by using the spectral decompositions, and using

$$S_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|), \quad S_x = \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|),$$

we have that (exercise)

$$\begin{aligned} e^{\frac{i}{\hbar}\theta S_z} S_x e^{-\frac{i}{\hbar}\theta S_z} &= \frac{\hbar}{2}(e^{i\theta}|+\rangle\langle-| + e^{-i\theta}|-\rangle\langle+|) \\ &= \cos\theta S_x - \sin\theta S_y. \end{aligned}$$

Note that this is exactly how the x-component of a vector transforms under a rotation about z. Because of this we get

$$\langle S_x\rangle \to \cos\theta\langle S_x\rangle - \sin\theta\langle S_y\rangle.$$

Similarly, it follows that

$$\langle S_y\rangle \to \cos\theta\langle S_y\rangle + \sin\theta\langle S_x\rangle,$$

and you can easily see that

$$\langle S_z\rangle \to \langle S_z\rangle$$

when the state vector is transformed. You can see that the unitary representative of rotations does indeed do its job as advertised.

Rotations represented on operators

Let us follow up on one result from above. We saw that

$$e^{\frac{i}{\hbar}\theta S_z} S_x e^{-\frac{i}{\hbar}\theta S_z} = \cos\theta S_x - \sin\theta S_y.$$

In the above equation the left hand side has the product of 3 Hilbert space operators appearing, corresponding to what happens to the spin vector observable when you change the state of the system via a unitary transformation corresponding to a rotation. The operator on the right hand side of the equation the linear combination of spin operators that you get by rotating them as if they are components of a vector in 3-d space.. This reflects a general rule which connects the rotations of 3-d space and their unitary representatives on the space of state vectors:

$$D^\dagger(\hat{n},\theta)\vec{S}D(\hat{n},\theta) = R(\hat{n},\theta)\vec{S}.$$

You can see immediately from this relation that the expectation values of $\vec{S}$ will behave like the components of a vector. More generally, if $\vec{V}$ is any trio of self-adjoint operators on Hilbert space representing a vector observable, then

$$D^\dagger(\hat{n},\theta)\vec{V}D(\hat{n},\theta) = R(\hat{n},\theta)\vec{V}.$$

You can think of this as analogous to the Heisenberg picture, but now for rotations. A rotation of the system can be mathematically viewed as a transformation of the state vector or, equivalently, as a transformation of the observables (but not both!).

Spin precession as a rotation

It is enlightening to return to the dynamical process of spin precession in light of our new results on rotations. You will recall that a spin system with magnetic moment $\vec{\mu}$ when placed in a uniform magnetic field $\vec{B}$ can be described by the Hamiltonian

$$H = -\vec{\mu} \cdot \vec{B}, \quad \text{where} \quad \vec{\mu} = \mu \vec{S}.$$

You will recall that the behavior of the spin observables could be viewed as precession about $\vec{B}$, *i.e.,* a continuously developing (in time) rotation about an axis along $\vec{B}$. We can now see this result immediately. Let $\hat{n}$ be a unit vector along $\vec{B}$, so that

$$H = -\mu B \hat{n} \cdot \mathbf{S}.$$

This means that the time evolution operator is

$$U(t, t_0) = e^{\frac{i}{\hbar}(t-t_0)\mu B \hat{n} \cdot \vec{S}}.$$

This operator represents a rotation about $\hat{n}$ by an "angle" $\mu B(t - t_0)$, which is exactly our previous result for the dynamics.

Note that, while the physical observables are precessing with frequency μB, the state vector itself has is precessing at half the frequency since, *e.g.,* it takes a 4π rotation to get the state vector to return to its initial value. For a single system we have seen that a 2π rotation changes the state vector by a phase so that there are no observable consequences of this mathematical fact. However, it is possible to experimentally "see" this difference in frequencies (thereby confirming the projective representation being used) by using a pair of spin 1/2 systems! We suppose that one of the spin 1/2 systems propagates freely and one travels through a region with a magnetic field. The latter spin will precess according to the time it spends in the magnetic field. The two particles can be brought together to form an interference pattern. The interference pattern depends upon the relative phases of the states of the two systems. If the magnetic field region is set up just right, you can arrange for the second particle to change its state vector by a minus sign by the time it leaves the magnetic field region. The interference pattern that you see confirms this fact. This effect has been observed experimentally! See your text for details.