Lecture 22

Relevant sections in text: §3.1

Infinitesimal Rotations and Angular Momentum

To represent rotations as transformations of quantum mechanical state vectors we need to find a way of continuously assigning to each rotation R a unitary transformation $D(R)$ on the Hilbert space such that

$$D(I) = I, \quad D(R_1)D(R_2) = e^{i\omega_{12}}D(R_1R_2),$$

for some choice of the real numbers ω.

Since $D(R)$ depends continuously upon an axis and angle, we can consider its infinitesimal form. For a fixed axis $\hat{n}$ and infinitesimal rotation angle ϵ we have

$$D(R) \approx I - \frac{i}{\hbar}\epsilon\hat{n}\cdot\vec{J},$$

where

$$\vec{J} = (J_1, J_2, J_3) = (J_x, J_y, J_z)$$

are self-adjoint operators, $J_i^\dagger = J_i$ with dimensions of angular momentum (in the sense that their matrix elements and eigenvalues have these dimensions). The operator J_i generates transformations on the Hilbert space corresponding to rotations of the system about the x^i axis. We identify the operators J_i with the angular momentum observables for the system. Of course, the physical justification of this mathematical model of angular momentum relies upon the unequivocal success of this strategy in describing physical systems. In particular, the J_i will (under appropriate circumstances) be conserved.

By demanding that the unitary transformations on the Hilbert space properly "mimic" (more precisely, "projectively represent") the rotations of 3-d space, it can be shown (see text for a version in which the phase factors are omitted) that the angular momentum operators satisfy the commutation relations

$$[J_k, J_l] = i\hbar\epsilon_{klm}J_m.$$

While the proof takes a little work and is omitted here, the result is very reasonable. Indeed, the commutation relations of infinitesimal generators in 3-d space encode the geometrical relationship between various rotations. It is therefore not surprising that the generators of rotations on the space of state vectors must obey the same commutation relations as the generators of rotations in 3-d space (up to the $i\hbar$, which is there because of the way we defined $\vec{J}$).

You have seen in your homework that the spin observables for a spin 1/2 system satisfy these commutation relations. Thus we identify the spin observables as a kind of angular momentum. This is not just a matter of terminology. In a closed system (*e.g.,* an atomic electron and a photon), angular momentum is conserved. However the angular momentum of a subsystem (*e.g.,* the electron) need not be conserved since it can exchange angular momentum with the rest of the system (*e.g.,* the photon) so long as the total angular momentum is conserved. The "bookkeeping" thus provided by conservation of angular momentum requires the spin angular momentum contribution to be included in order to "balance the books". Spin angular momentum provides a contribution to the conserved angular momentum of a closed system.

Using the same mathematical technology as we did for time and space translations, it is not hard to see that a finite (as opposed to infinitesimal) rotation can formally be built out of "many" infinitesimal rotations, leading to the formula:

$$D(\hat{n}, \theta) = \lim_{N \to \infty} (I - \frac{i}{\hbar} \frac{\theta}{N} \hat{n} \cdot \vec{J}) = e^{-\frac{i}{\hbar} \theta \hat{n} \cdot \vec{J}}.$$

The detailed form of this exponential operator, like that of the J_i depends upon the specific physical system being studied. The most familiar form of angular momentum is probably that of a particle moving in 3-d. However, spin also is a form of angular momentum (according to the above type of analysis) and it is the simplest, mathematically speaking, so we shall look at it first.