Lecture 21

Relevant sections in text: §3.1

Angular momentum - introductory remarks

The theory of angular momentum in quantum mechanics is important in many ways. The myriad of results of this theory, which follow from a few simple principles, are used extensively in applications of quantum mechanics to atomic, molecular, nuclear and other subatomic systems. The mathematical strategies involved have a number of important generalizations to other types of symmetries and conservation laws in quantum mechanics. The quantum mechanical theory of angular momentum leads naturally to the phenomenon of "intrinsic spin". Just as we saw for spin 1/2, a general feature of angular momentum in quantum mechanics is the incompatibility of the observables corresponding to any two components of angular momentum. The nature of this incompatibility is at the heart of virtually all features of angular momentum.

Just as linear momentum is intimately connected with the notion of translation of the physical system, so angular momentum is deeply tied to the theory of rotations of the physical system being considered. We shall use this geometric interpretation of angular momentum as the starting point for our discussion.

Rotations in three dimensions

We now begin our discussion of angular momentum using its geometric interpretation as the generator of rotations in space. I should emphasize at the outset that our discussion can be a little confusing because we will be studying vectors and linear transformations in 2 distinct spaces: (i) the 3-d (Euclidean) space we live in, and (ii) the Hilbert space of quantum state vectors. The 3-d rotations are, of course, going to be related to corresponding transformations on the space of quantum states, but it is not too hard to get mixed up about which space various quantities are associated with. So watch out!

We begin by summarizing some elementary results concerning rotations in three dimensions. *This part of the discussion is completely independent of quantum mechanical considerations.* Until you are otherwise notified, everything we do will only refer to properties of rotations of observables in the 3-d space live in.

A vector observable for some physical system, $\vec{V}$, responds to a rotation according to a (special) orthogonal transformation:

$$\vec{V} \to R\vec{V}.$$

Here R is a linear transformation of 3-d vectors such that

$$(R\vec{V}) \cdot (R\vec{W}) = \vec{V} \cdot \vec{W}.$$

Evidently, magnitudes of vectors as well as their relative angles are invariant under this transformation.

If you represent vectors $\vec{V}$, $\vec{W}$ as column vectors V, W relative to some Cartesian basis, the dot product is

$$\vec{V} \cdot \vec{W} = V^T W = W^T V.$$

You can then represent R as a 3×3 matrix, also denoted R for convenience, acting on the (Cartesian) components of $\vec{V}$ and satisfying (exercise)

$$R^T = R^{-1},$$

that is

$$R^T R = I = R R^T,$$

where the superscript T means "transpose" and I is the 3×3 identity matrix. This requirement guarantees the invariance of dot products:

$$V^T W \longrightarrow (RV)^T (RW) = V^T R^T R W = V^T W.$$

Such matrices are called *orthogonal* (do you know why?). As a simple example, a rotation about the z axis by an angle θ is represented by

$$R(\hat{z}, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The inverse of this matrix is easily verified to be the transpose:

$$R^{-1}(\hat{z}, \theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = R(\hat{z}, -\theta).$$

In general, rotations are defined relative to an origin, which is fixed by any rotation. The rotation is then defined by giving an *axis* of rotation through the origin and an angle of rotation about that axis. The axis itself can be specified by a unit vector $\hat{n}$. We will write $R(\hat{n}, \theta)$ for the orthogonal transformation so-defined by the axis through the origin along $\hat{n}$ and by the angle θ. The sense of the rotation (counterclockwise or clockwise) is determined by the right-hand rule. For any single rotation we can always choose the z-axis to be along $\hat{n}$ and then the rotation matrix takes the form given above. Of course, when considering different rotations about different axes one cannot put them all into this simple form. You can see that it takes 3 numbers (two for $\hat{n}$ and one for θ) to specify a rotation. The set of all rotations about a point forms a three-dimensional group (since it has 3 parameters). This means, in particular, that every rotation has an inverse, and that the product of two rotations is equivalent to third rotation. This group is called the *rotation group* and denoted by $SO(3)$. The "3" means "3 dimensions". The "O" means "orthogonal". And the "S" means "special". This latter adjective arises since not all orthogonal transformations are rotations, they also include discrete transformations: reflections and inversions. All orthogonal matrices have determinants of ± 1. To see this, recall that $\det(AB) = \det(A)\det(B)$ and $\det A^T = \det A$, so that

$$OO^T = I \Longrightarrow [\det(O)]^2 = 1.$$

The rotations are described by matrices with unit determinant, while the discrete transformations (that are not rotations in disguise) have negative determinant. For example, the transformation

$$\vec{V} \to -\vec{V}$$

is given by the 3×3 orthogonal matrix $O = -I$, which has determinant -1. The rotation group is *non-Abelian*, which means "non-commutative", since successive rotations commute if and only if they are about the same axis. The ways in which successive rotations combine to make a third rotation is somewhat intricate. However, this complicated behavior can be fruitfully analyzed by studying infinitesimal rotations.

Infinitesimal Rotations

Our goal is to view angular momentum as the infinitesimal generator of rotations *on the space of quantum states*, so we need to understand rotations from the infinitesimal point of view in ordinary 3-d space. Since rotations depend continuously on the angle of rotation, we can consider rotations that are "infinitesimal", that is, nearly the identity. An infinitesimal rotation *in 3-d space* about an axis $\hat{n}$ and angle $\epsilon << 1$ can be written as

$$R(\hat{n}, \epsilon) \approx I + \epsilon G,$$

where the linear transformation G is the *generator* of rotations— it can be viewed as a 3×3 matrix—and we are ignoring terms of order ϵ^2. (I emphasize that, we are presently considering rotations in 3-d space; we haven't yet moved to the representation of rotations on state vectors.) Note that if $R(\hat{n}, \epsilon)$ is to be orthogonal then G must be an antisymmetric matrix:

$$G^T = -G.$$

To see this just examine the relation

$$I = [I + \epsilon G + \mathcal{O}(\epsilon^2)]^T [I + \epsilon G + \mathcal{O}(\epsilon^2)].$$

For example, if $\hat{n}$ is along z we have that (exercise)

$$G_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can define a rotation generator for rotation about any axis. We write

$$R(\hat{n}, \epsilon) \approx I + \epsilon \hat{n} \cdot \vec{G},$$

where

$$\vec{G} = (G_1, G_2, G_3) = (G_x, G_y, G_z)$$

are a basis for the 3-dimensional vector space of anti-symmetric matrices. G_z is displayed above; you can easily compute the forms for G_x and G_y by expanding the rotation matrices about the indicated axes to first order in the rotation angle (exercise):

$$G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

A straightforward computation reveals (exercise):

$$[G_i, G_j] = \epsilon_{ijk} G_k.$$

These are the *commutation relations* of infinitesimal rotations. They give a complete (albeit infinitesimal) account of the way in which successive rotations combine to give a net rotation. In essence, the generators and their commutation relations define the group of rotations. Indeed, just as with translations, we can build up a finite rotation about an axis along $\hat{n}$ by an infinite number of infinitesimal rotations according to

$$R(\hat{n}, \theta) = \lim_{N \to \infty} (I + \frac{\theta}{N} \hat{n} \cdot G)^N = e^{\theta \hat{n} \cdot G}.$$

The commutation relations encode the relationships between different rotations.

Note that these commutation relations, which are for the generators of rotations in 3-d space, look a lot like those you encountered for the components of the spin operators. Of course, these vector observables constitute operators on a 2-d Hilbert space. But the similarity in the commutation relations is no accident as we shall see.

Rotations in quantum mechanics

Now we will discuss what the preceding considerations have to do with quantum mechanics. In quantum mechanics transformations in space and time are "implemented" or "represented" by unitary transformations on the Hilbert space for the system. The idea is that if you apply some transformation to a physical system in 3-d, the state of the system is changed and this should be mathematically represented as a transformation of the state vector for the system. We have already seen how time translations and spatial translations are described in this fashion. Following this same pattern, to each rotation

R we want to define a unitary transformation, $D(R)$, such that if $|\psi\rangle$ is the state vector for the system, then $D(R)|\psi\rangle$ represents the state vector after the system has undergone a rotation characterized by R. The key requirement here is that the pattern for combining two rotations to make a third rotation is "mimicked" by the unitary operators. For this we require that the unitary operators $D(R)$ depend continuously upon the rotation axis and angle and satisfy

$$D(R_1)D(R_2) = e^{i\omega_{12}}D(R_1R_2),$$

where ω_{12} is a real number, which may depend upon the choice of rotations R_1 and R_2, as its notation suggests. This phase freedom is allowed since the state vector $D(R_1R_2)|\psi\rangle$ cannot be *physically* distinguished from $e^{i\omega_{12}}D(R_1R_2)|\psi\rangle$.

If we succeed in constructing this family of unitary operators $D(R)$, we say we have constructed a "unitary representation of the rotation group up to a phase", or a "projective unitary representation of the rotation group". You can think of all the ω parameters as simply specifying, in part, some of the freedom one has in building the unitary representatives of rotations. (If the representation has all the ω parameters vanishing we speak simply of a "unitary representation of the rotation group".)

This possible phase freedom in the combination rule for representatives of rotations is a purely quantum mechanical possibility and has important physical consequences. Incidentally, your text book fails to allow for this phase freedom in the general definition of representation of rotations. This is a pedagogical error, and an important one at that. This error is quite ironic: the first example the text gives of the $D(R)$ operators is for a spin 1/2 system where the phase factors are definitely non-trivial, as we shall see.