Lecture 19

Relevant sections in text: §2.3, 2.6

Oscillator Dynamics – Heisenberg picture

Time dependent features of the oscillator (as with any system) appear in non-stationary states. Let us examine this using the Heisenberg picture. The principal job is to get the basic observables at time t in terms of their representation at some fixed initial time, say, $t_0 = 0$. We have

$$X(0) \equiv X, \quad P(0) \equiv P.$$

We need to compute $X(t)$ and $P(t)$. We can do this directly using

$$X(t) = e^{\frac{i}{\hbar}Ht}X(0)e^{-\frac{i}{\hbar}Ht}, \quad P(t) = e^{\frac{i}{\hbar}Ht}P(0)e^{-\frac{i}{\hbar}Ht},$$

and

$$H = H(t) = \frac{P^2(t)}{2m} + \frac{1}{2}m\omega^2 X^2(t) = H(0) = \frac{P^2(0)}{2m} + \frac{1}{2}m\omega^2 X^2(0).$$

To evaluate $X(t)$ and $P(t)$ you can either expand the exponentials in power series and, by studying the general term, try to deduce a closed form expression for the result. There are other tricks as well for manipulating the similarity transformation. However, a somewhat easier way to get expressions for $X(t)$ and $P(t)$ is to directly solve the Heisenberg equations. Using (exercise)

$$[X(t), P(t)] = i\hbar I,$$

we have

$$i\hbar\frac{d}{dt}X(t) = [X(t), H] = i\hbar\frac{P(t)}{m},$$

$$i\hbar\frac{d}{dt}P(t) = [P(t), H] = -i\hbar m\omega^2 X(t),$$

So that the Heisenberg equations can be written as

$$\frac{d}{dt}X(t) = \frac{P(t)}{m}, \quad \frac{d}{dt}P(t) = -m\omega^2 X(t).$$

These are formally the same as the classical Hamilton equations of motion, and are as easily solved. Taking account the initial conditions at $t = 0$, we have

$$X(t) = (\cos\omega t)X(0) + (\frac{1}{m\omega}\sin\omega t)P(0),$$

$$P(t) = (\cos\omega t)P(0) - (m\omega\sin\omega t)X(0).$$

In the Heisenberg picture, the time dependence in physical predictions (*i.e.,* probability distributions) is obtained using these observables and a fixed state vector. As an example, let us compute the statistical mean of the position and momentum at time t. Taking diagonal matrix elements of the Heisenberg operators, we get

$$\langle X\rangle(t) = (\cos\omega t)\langle X\rangle(0) + (\frac{1}{m\omega}\sin\omega t)\langle P\rangle(0),$$

$$\langle P\rangle(t) = (\cos\omega t)\langle P\rangle(0) - (m\omega\sin\omega t)\langle X\rangle(0).$$

From the above equations you can see that we get the usual sort of behavior expected of a harmonic oscillator with mass m and frequency ω. Indeed, we see that the expectation values behave exactly as do the classical position and momentum, in accord with Ehrenfest's theorem.

Of course, if the system is in a stationary state $|n\rangle$ then expectation values do not change in time. This seems paradoxical in light of our result shown above. The resolution of this paradox in the above example is that, for stationary states, the initial expectation values vanish, *e.g.,*

$$\langle X\rangle(t) = \langle X\rangle(0) = 0, \quad \langle P\rangle(t) = \langle P\rangle(0) = 0.$$

As an exercise see if you can show how the expectation values of, say, X^2 manages to be time independent in a stationary state.

Charged particle in an electromagnetic field

We now turn to another extremely important example of quantum dynamics. Let us describe a non-relativistic particle with mass m and electric charge q moving in a given electromagnetic field. This system has obvious physical significance.

We use the same position and momentum operators (in the Schrödinger picture) $\vec{X}$ and $\vec{P}$. To describe the electromagnetic field we need to use the electromagnetic scalar and vector potentials $\phi(\vec{x},t)$, $\vec{A}(\vec{x},t)$. They are related to the familiar electric and magnetic fields $(\vec{E},\vec{B})$ by

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \nabla\times\vec{A}.$$

The dynamics of a particle with mass m and charge q is determined by the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}(\vec{X},t)\right)^2 + q\phi(\vec{X},t).$$

This Hamiltonian takes the same form as the classical expression in Hamiltonian mechanics. We can see that this is a reasonable form for H by computing the Heisenberg equations of motion, and seeing that they are equivalent to the Lorentz force law, which we shall now demonstrate.

Some typical electromagnetic potentials that are considered are the following.

(i) The Coulomb field, with

$$\phi = \frac{k}{|\vec{x}|}, \quad \vec{A} = 0,$$

which features in a simple model of the hydrogen atom; the spectrum and stationary states should be familiar to you. We will soon study it a bit in the context of angular momentum issues.

(ii) A uniform magnetic field $\vec{B}$, where

$$\phi = 0, \quad \vec{A} = \frac{1}{2} B \times \vec{x}.$$

The vector potential is not unique, of course. This potential is in the Coulomb gauge. You will explore this system in your homework. The results for the stationary states are interesting. One has a continuous spectrum coming from the motion along the magnetic field; but for a given momentum value there is a discrete spectrum of "Landau levels" coming from motion in the plane orthogonal to $\vec{B}$. To see this one massages the Hamiltonian into the mathematical form of a free particle in one dimension added to a harmonic oscillator; this is the gist of your homework problem.

(iii) An electromagnetic plane wave, in which

$$\phi = 0, \quad \vec{A} = \vec{A}_0 \cos(\vec{k} \cdot \vec{x} - kct), \quad \vec{k} \cdot \vec{A}_0 = 0.$$

Of course, this latter example involves a time dependent potential. This potential is used to study the very important issue of interaction of electrons with a radiation field. If we use the Coulomb potential for ϕ then this would give a model for an atomic electron in a radiation field; we should have time to study this toward the end of the semester.

For simplicity we will henceforth assume that the potentials are time independent — then the Heisenberg and Schrödinger picture Hamiltonians are the same:

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}(\vec{X})\right)^2 + q\phi(\vec{X}).$$

For the positions we get (exercise)

$$\frac{d}{dt}\vec{X}(t) = \frac{1}{i\hbar}[\vec{X}(t), H] = \frac{1}{m}\{\vec{P}(t) - \frac{q}{c}\vec{A}(\vec{X}(t))\}.$$

We see that (just as in classical mechanics) the momentum – defined as the generator of translations – is not necessarily given by the mass times the velocity, but rather

$$\vec{P}(t) = m\frac{d\vec{X}(t)}{dt} + \frac{q}{c}\vec{A}(\vec{X}(t)).$$

As in classical mechanics we sometimes call $\vec{P}$ the *canonical momentum*, to distinguish it from the *mechanical momentum*

$$\vec{\Pi} = m\frac{d\vec{X}(t)}{dt} = \vec{P} - \frac{q}{c}\vec{A}(\vec{X}(t))$$

Note that the mechanical momentum has a direct physical meaning, while the canonical momentum has a direct mathematical meaning but depends upon the non-unique form of the potentials. We will discuss this in detail soon.