Lecture 16

Relevant sections in text: §2.2

Example: Free particle in 1-d

Let us revisit the free particle in 1-d using the Heisenberg equations. The Hamiltonian is the same operator in either picture:

$$H = \frac{P^2}{2m} = \frac{P^2(t)}{2m}.$$

This is easily seen since

$$[\frac{P^2}{2m}, e^{-\frac{i}{\hbar}\frac{P^2}{2m}}] = 0.$$

Next we have

$$[X(t), P(t)] = i\hbar I,$$

so

$$[X(t), P^2(t)] = 2i\hbar P(t).$$

Obviously,

$$[P(t), P^2(t)] = 0,$$

so the Heisenberg equations are

$$\frac{dX(t)}{dt} = \frac{P(t)}{m}, \quad \frac{dP(t)}{dt} = 0,$$

with solution

$$X(t) = X + \frac{P}{m}, \quad P(t) = P.$$

Some other features of the Heisenberg picture

The Heisenberg equation can make certain results from the Schrödinger picture quite transparent. For example, just by taking expectation values on both sides of the equation using the (single, time-independent) state vector it is apparent that (exercise)

$$\frac{d}{dt}\langle \mathbf{A}\rangle(t) = \frac{1}{i\hbar}\langle [A, H]\rangle.$$

Here the notation $\frac{1}{i\hbar}[A, H]$ means the observable corresponding to the commutator, which is well defined in either picture:

$$\langle [A, H]\rangle = \langle \psi, t|[A, H]|\psi, t\rangle = \langle \psi, t_0|[A(t), H_{Heis}(t)]|\psi, t_0\rangle.$$

Similarly, assuming for simplicity that $H = H_{Heis}$ is time independent, it is easy to see that operators which commute with the Hamiltonian at any one time, say t_0, are *constants of the motion*:

$$\frac{d}{dt}A(t) = \frac{1}{i\hbar}[A(t), H] = U^\dagger[A(t_0), H]U = 0.$$

Thus, conserved quantities satisfy

$$A(t) = A(t_0) = A.$$

You can also see this result directly. If an observable commutes with H at one time, $t = t_0$ say, then it will not change in time since

$$A(t) = U^\dagger(t, t_0)A(t_0)U(t, t_0) = U^\dagger(t, t_0)U(t, t_0)A(t_0) = A(t_0).$$

Evidently, a conserved quantity has a time independent probability distribution. You can see this by noting that, since the operator in the Heisenberg picture does not change in time, its eigenvectors are also time independent so that the scalar product of eigenvectors with state vectors is time independent.

We have seen that the time evolution operator defines, via a unitary/similarity transformation, the Heisenberg operators at time t:

$$A(t) = U^\dagger AU.$$

If we have an observable that is a function of A, $F(A)$ say, we have, of course

$$F(A)(t) = U^\dagger F(A)U.$$

It is important to note that one can also express this as

$$F(A)(t) = F(A(t)) \equiv F(U^\dagger AU).$$

To see this, we use the spectral decomposition:

$$F(A) = \sum_a F(a)|a\rangle\langle a|,$$

so that

$$U^\dagger F(A)U = \sum_a F(a)U^\dagger|a\rangle\langle a|U = \sum_a F(a)|a(t)\rangle\langle a(t)| = F(A(t)).$$

Heisenberg Picture Dynamics of a Particle in a potential

For a particle with position (at one time, say $t = 0$) $\vec{X}$ and momentum $\vec{P}$, we consider a Hamiltonian of the form

$$H = \frac{P^2}{2m} + V(\vec{X}).$$

Note that this operator does not depend upon time, so it is both the Schrödinger and Heisenberg Hamiltonian. In particular, we have that

$$H(t) = H(0) = H = \frac{P(t)^2}{2m} + V(\vec{X}(t)).$$

This result can be viewed as a mathematical version of the conservation of energy in the Heisenberg picture.

Let us compute the Heisenberg equations for $\vec{X}(t)$ and momentum $\vec{P}(t)$. Evidently, to do this we will need the commutators of the position and momentum with the Hamiltonian. To begin, let us consider the canonical commutation relations (CCR) at a fixed time in the Heisenberg picture. Using the general identity

$$[A(t), B(t)] = U^\dagger(t, t_0)[A(t_0), B(t_0)]U(t, t_0),$$

we get (exercise)

$$[X^i(t), X^j(t)] = 0 = [P_i(t), P_j(t)], \quad [X^i(t), P_j(t)] = i\hbar\delta^i_j I.$$

In other words, the Heisenberg position and momentum operators obey the CCR at any fixed time. Note that this means the uncertainty relation between position and momentum is time independent — a fact you can also prove in the Schrödinger picture (exercise).

It is now straightforward to compute

$$[X^i(t), H] = \frac{i\hbar}{m} P^i(t),$$

so that

$$\frac{dX^i(t)}{dt} = \frac{1}{m} P^i(t).$$

Thus one relates the momentum and velocity operators of the particle; a result that is a bit more tricky to establish in the Schrödinger picture. To compute the Heisenberg equations for the momentum we need to compute (exercise)

$$[P_i(t), H] = [P_i(t), V(\vec{X}(t))].$$

Probably the simplest way to do this is to use

$$[P_i(t), V(\vec{X}(t))] = U^\dagger(t, 0)[P_i, V(\vec{X})]U(t, 0) = -i\hbar U^\dagger(t, 0)\frac{\partial V}{\partial x^i}(\vec{X})U(t, 0) = -i\hbar\frac{\partial V}{\partial x^i}(\vec{X}(t)).$$

Here we have used

$$[P_i, V(\vec{X})] = -i\hbar\frac{\partial V}{\partial x^i}(\vec{X}),$$

which can be verified by checking it on the position eigenvector basis using the definition of P_i as the generator of infinitesimal translations (good exercise). All together we get

$$\frac{dP_i(t)}{dt} = -\frac{\partial V}{\partial x^i}(\vec{X}(t)).$$

Using the Heisenberg equations for $\vec{X}(t)$, we can write the Heisenberg equation for $\vec{P}(t)$ as

$$\frac{d^2 X^i(t)}{dt^2} = F^i(\vec{X}(t)),$$

where

$$F^i(\vec{X}(t)) = -\frac{\partial V}{\partial x^i}(\vec{X}(t)),$$

can be viewed as the quantum representation of the force at time t in the Heisenberg picture. This is a quantum version of Newton's second law.

From this result it is tempting to believe that a quantum particle is actually behaving just like a classical particle. After all, the basic observables obey the same equations of motion in the two theories. Of course this is not true, if only because it is not possible to know both position and momentum with statistical certainty in the quantum theory. In the next section we will take a closer look at this issue.