Lecture 15

Relevant sections in text: §2.2

More on the Heisenberg picture

As you know, one of the most basic predictions of quantum mechanics is that the set of possible outcomes of a measurement of **A** is the spectrum of its operator representative A. In the Schrödinger picture, we consider the spectrum of A to get at the possible outcomes of a measurement of **A** at any time. In the Heisenberg picture we have to consider a different operator $A(t)$ at each time to see what are the possible outcomes of a measurement of **A** at that time. This looks bad. If in the Heisenberg picture the operator representing **A** can be different at different times, we are in danger of saying that the possible outcomes of a measurement of **A** – which is a fixed set for all time in the Schrödinger picture – is different at different times. For example, *perhaps* it is possible that a particle could, at one instant of time, have the whole real line to move on while it could only move on some subset of the real line at some other time. This would be a serious inconsistency. Of course, there is no inconsistency. You can easily check that if $|a_i\rangle$ are the eigenvectors of A,

$$A|a_i\rangle = a_i|a_i\rangle,$$

then

$$|a_i, t\rangle := U^\dagger(t, t_0)|a_i\rangle$$

are eigenvectors of $A(t)$ *with the same eigenvalue*:

$$A(t)|a_i, t\rangle = a_i|a_i, t\rangle.$$

The spectrum of $A(t)$ is identical to the spectrum of A.* Thus the possible outcomes of a measurement of **A** are the same in both pictures.

What *does* change between the two pictures is the mathematical representation of the states in which the observable **A** is fixed with statistical certainty. As you know, the states where **A** are known with certainty are the eigenvectors of its operator representative. In the Heisenberg picture we then get, in general, a different eigenvector at each time. This is exactly what is needed to get the proper time evolution of probability distributions:

$$Prob(\mathbf{A} = a_i, t) = |\langle a_i, t|\psi, t_0\rangle|^2 = |\langle a_i|U(t, t_0)|\psi, t_0\rangle|^2 = |\langle a_i|\psi, t|\rangle|^2,$$

where the last equality gives the Schrödinger picture formula for the probability.

This result on the eigenvectors changing in time can lead to confusion, so let me belabor the point a bit. The state vector in the Heisenberg picture is the same for all time, but

* This is true for any two operators, A, A', related by a unitary transformation: $A' = U^\dagger A U$.

the (basis of) eigenvectors of an observable will, in general, change in time. If you start your system off in an eigenvector of $A(t_0)$, this means that the observable **A** is known with certainty to have the value, say a, at time $t = t_0$ and we have $|\psi, t\rangle = |\psi, t_0\rangle = |a, t_0\rangle$. At some other time t, the state vector is still $|a, t_0\rangle$, but this is no longer a state in which **A** is known with certainty since that state vector would be $|a, t\rangle$, not $|a, t_0\rangle$. So, for example, if you know with certainty that the observable **A** has the value a at time t_0, then at time t the probability for getting a is

$$|\langle a, t|a, t_0\rangle|^2 = |\langle a, t_0|U(t, t_0)|a, t_0\rangle|^2.$$

One sometimes summarizes this situation with the slogan: In the Schrödinger picture the basis vectors (provided by the observables) are fixed, while the state vector evolves in time. In the Heisenberg picture the state vectors are held fixed, but the basis vectors evolve in time (in the inverse manner). This is an instance of the "active vs. passive" representation of a transformation — in this case time evolution.

Finally, let us check that the physically important notion of compatible/incompatible observables is left undisturbed by the transition to the Heisenberg picture. Let us compare the commutators of the operator/observables in each picture.

$$[A(t), B(t)] = [U^\dagger AU, U^\dagger BU] = U^\dagger [A, B]U.$$

(Note any two operators related by a unitary/similarity transformation will have this commutator property.) Thus, if **A** and **B** are (in)compatible in the Schrödinger picture they will be (in)compatible (at each time) in the Heisenberg picture. Note also that the commutator of two observables in the Schrödinger picture — which is i times another observable, say, C, — makes the transition to the Heisenberg picture just as any other Schrödinger observable, namely, via the unitary transformation

$$C \to U^\dagger CU.$$

Unitary transformations in general

Let us note that while our discussion was phrased in the context of time evolution, the same logic can be applied to any unitary transformation. For example, for a particle moving in one dimension one can view the effect of translations as either redefining the state vector, leaving the operator-observables unchanged:

$$|\psi\rangle \to T_a|\psi\rangle, \quad A \to A$$

or equivalently as redefining the observables, with the state vectors unchanged:

$$A \to T_a^\dagger A T_a, \quad |\psi\rangle \to |\psi\rangle.$$

Note, in particular, that the position and momentum operators change in the expected way under a translation (exercise – you played with this stuff in the homework):

$$T_a^\dagger X T_a = X + aI, \quad T_a^\dagger P T_a = P.$$

Heisenberg equations

We saw that the conventional Schrödinger equation is really just a consequence of the relation between the time evolution operator and its infinitesimal generator in the context of the Schrödinger picture:

$$i\hbar\frac{d}{dt}U(t,t_0) = H(t)U(t,t_0) \quad \Longleftrightarrow \quad i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle.$$

Given a Hamiltonian, this equation is the starting point for investigating quantum dynamics in the Schrödinger picture. We can now ask: what is the analog of this in the Heisenberg picture? In the Heisenberg picture, dynamical evolution occurs through the operator-observables, which for simplicity we assume to be time independent in the Schrödinger picture. We have

$$A(t) = U^\dagger(t,t_0)AU(t,t_0).$$

Differentiating both sides and using our basic differential equation for $U(t,t_0)$ we get

$$\begin{aligned}\frac{d}{dt}A(t) &= -\frac{1}{i\hbar}U^\dagger(t,t_0)H(t)AU(t,t_0) + U^\dagger(t,t_0)A\frac{1}{i\hbar}H(t)U(t,t_0)\\ &= -\frac{1}{i\hbar}U^\dagger(t,t_0)H(t)U(t,t_0)U^\dagger(t,t_0)AU(t,t_0) + \frac{1}{i\hbar}U^\dagger(t,t_0)AU(t,t_0)U^\dagger(t,t_0)H(t)U(t,t_0)\\ &= \frac{1}{i\hbar}[A(t), H_{Heis}(t)].\end{aligned}$$

Here we have introduced the Heisenberg picture version of the Hamiltonian:

$$H_{Heis}(t) = U^\dagger(t,t_0)H(t)U(t,t_0).$$

If the (Schrödinger) Hamiltonian is time independent, as is often the case, then we have (exercise)

$$U(t,t_0) = \exp\left\{-\frac{i}{\hbar}(t-t_0)H\right\}.$$

Thus U is a function of a single operator, the Hamiltonian. Because a function of an operator will always commute with that operator we have

$$HU(t,t_0) = U(t,t_0)H,$$

so that

$$H_{Heis}(t) = H.$$

This is pretty important to keep in mind; again, it only works when $\frac{\partial H}{\partial t} = 0$.

The equation

$$\frac{d}{dt}A(t) = \frac{1}{i\hbar}[A(t), H_{Heis}(t)]$$

is the *Heisenberg equation of motion* for the Heisenberg operator $A(t)$. Given a Hamiltonian it is a differential equation that, in principle, can be solved to find the Heisenberg operator corresponding to an observable at time t, given initial conditions

$$A(t_0) = A.$$

So, given a Hamiltonian H_{Heis}, to analyze dynamics in the Heisenberg picture one solves the Heisenberg equations for the observable(s) of interest. Given the solution, say $A(t)$, one gets the time dependence of probability distributions in the usual way. The outcome of a measurement of the observable represented (at time t) by $A(t)$ is one of the eigenvalues a_i. The probability for getting a_i (assuming no degeneracy) at time t is

$$P(a_i, t) = |\langle a_i, t|\psi, t_0\rangle|^2 = |\langle a_i, t_0|U(t, t_0)|\psi, t_0\rangle|^2,$$

where $|a_i, t\rangle$ is the eigenvector of $A(t)$ with eigenvalue a_i and $|\psi, t_0\rangle$ is the state vector for the system.