Lecture 14

Relevant sections in text: §2.1, 2.2

Time-Energy Uncertainty Principle

We have seen that (for a time-independent Hamiltonian) energy eigenvectors define stationary states, so that if the energy is known with certainty at one time, then the state is unchanged for all time, *i.e., all* probability distributions are time independent. Physical attributes of the system will evolve in time only if the initial state of the system is a superposition of energy eigenvectors with different energies. Of course, in such an initial state the energy is not known "with certainty", but rather has a non-trivial probability distribution. Evidently, a the statistical uncertainty in energy is related to the time rate of change of (probability distributions of) observables. The infamous *time-energy uncertainty principle* relates the time scale Δt for a significant change of a given (initial) state to the statistical uncertainty of the energy ΔE in that state (other misleading slogans notwithstanding). You will see that the meaning of this uncertainty principle is a little different than, *e.g.,* the position-momentum uncertainty principle, though it does take the same form

$$\Delta t \Delta E \sim \hbar.$$

Keep in mind that the time is not viewed as an observable to be represented as a self-adjoint operator, so the general form of uncertainty relation for incompatible observables does not apply here.

Suppose the energy is discrete, for simplicity, with values E_k and eigenvectors $|k\rangle$. Any state can be written as

$$|\psi\rangle = \sum_k c_k |k\rangle.$$

Assuming this is the initial state at $t = t_0$, the state at time t is given by

$$U(t, t_0)|\psi\rangle = \sum_k c_k e^{-\frac{i}{\hbar}E_k(t-t_0)}|E_k\rangle.$$

Let us monitor the time evolution in the physical system in terms of the time variation of an observable A (which is, after all, what we actually do). Let us denote the standard deviation of A (or H) in the initial state $|\psi\rangle$ by ΔA (or ΔE). From the uncertainty relation we have in the initial state

$$\Delta A \Delta E \geq \frac{1}{2}|\langle [A, H]\rangle|.$$

Recall our previous result which relates time evolution of expectation values to commutators; we get

$$\frac{1}{2}|\langle [A, H]\rangle| = \frac{\hbar}{2}|\frac{d}{dt}\langle A\rangle|.$$

Therefore:

$$\Delta A \Delta E \geq \frac{\hbar}{2} \left| \frac{d}{dt} \langle A \rangle \right|.$$

Let us use Δt to represent the time scale over which the system changes significantly from the initial state. We can do this by comparing the rate of change of the average value of A to the initial uncertainty in A:

$$\Delta t = \frac{\Delta A}{\left|\frac{d}{dt}\langle A \rangle\right|}.$$

With Δt *so-defined* we then have

$$\Delta t \Delta E \geq \frac{\hbar}{2}.$$

So, the shortest possible time scale that characterizes a significant change in the system is given by

$$\Delta t \Delta E \sim \hbar.$$

Of course, if the (initial) state is stationary – that is, an energy eigenvector, then $\Delta E = 0$, which forces $\Delta t \to \infty$, which makes sense since the physical attributes of the state never change.

The time-energy uncertainty principle is then a statement about how the statistical uncertainty in the energy (which doesn't change in time since the energy probability distribution doesn't change in time) controls the time scale for a change in the system. In various special circumstances this fundamental meaning of the time-energy uncertainty principle can be given other interpretations, but they are not as general as the one we have given here. Indeed, outside of these "special circumstances", the alternative interpretations of the time-energy uncertainty principle can become ludicrous. What I am speaking of here are things like the oft-heard "You can violate conservation of energy if you do it for a short enough time", or "The uncertainty of energy is related to the uncertainty in time". We shall come back to these bizarre sounding statements – which have good meaning in various specialized contexts – and see what they really mean a little bit later. For now, beware of such slogans.

As a nice example of the time-energy uncertainty principle, consider the spin precession problem we just studied. Recall that we had a uniform, static magnetic field $\mathbf{B} = B\hat{k}$ along the z axis. The Hamiltonian is

$$H = \frac{eB}{mc} S_z.$$

We studied the time dependence of the spin observables when the initial state was an S_x eigenvector. It is not hard to compute the standard deviation of energy in the initial state $|S_x, +\rangle$. Using

$$H^2 = \left(\frac{eB\hbar}{2mc}\right)^2 I,$$

we have (exercise)

$$\Delta E = \frac{\hbar}{2}\frac{eB}{mc} = \frac{1}{2}\hbar\omega,$$

so that we expect a significant change in the state when

$$\frac{1}{2}\omega\Delta t \sim 1.$$

Thus the frequency ω controls the time scale for changes in the system, as you might have already ascertained from, *e.g.,* the probability distributions

$$Prob(S_x = \pm\frac{\hbar}{2})(t) = \begin{cases} \cos^2(\frac{\omega}{2}t) \\ \sin^2(\frac{\omega}{2}t) \end{cases}.$$

Heisenberg picture

Let us now see how to describe dynamics using the Heisenberg picture, in which we encode the time evolution into the operator representatives of the observables rather than in the state vectors. The idea is that time evolution is mathematically modeled by allowing the correspondence between physical observables and self-adjoint operators to change in time. To see how this works is straightforward, given our previous work in the Schrödinger picture.

It is useful to use a very explicit notation for a while. We denote the physical observable by $\mathbf{A}$. We assume $\mathbf{A}$ is defined in a time independent fashion.* We denote its time-independent (in the Schrödinger picture), self-adjoint operator representative by A. We have

$$\langle \mathbf{A}\rangle(t) = \langle\psi, t|A|\psi, t\rangle = \langle\psi, t_0|U^\dagger(t, t_0)AU(t, t_0)|\psi, t_0\rangle.$$

Keep in mind that all physical predictions can be computed via expectation values. If we *define*

$$A(t) = U^\dagger(t, t_0)AU(t, t_0),$$

then we can view time evolution as occurring, mathematically speaking, through the time dependent identification of a self-adjoint operator $A(t)$ to the observable $\mathbf{A}$:

$$\langle \mathbf{A}\rangle(t) = \langle\psi, t_0|A(t)|\psi, t_0\rangle.$$

* For simplicity, we shall always restrict our attention to observables $\mathbf{A}$ which do not involve the time t explicitly in their definition. What I mean here is that we shall exclude from our discussion observables like "the energy multiplied by the time". It is easy to allow for such observables, but it might be confusing at first glance. Your text shows how to generalize our results to the case of "explicitly time dependent observables".

We call A the Schrödinger picture operator/observable and we call $A(t)$ the Heisenberg picture operator/observable. You can see that at $t = t_0$ the Schrödinger and Heisenberg representatives of $\mathbf{A}$ are the same.

In the Heisenberg picture the unit vector representing the state of the system is fixed once and for all,

$$|\psi, t\rangle = |\psi, t_0\rangle,$$

while the operator-observables evolve in time. In the Schrödinger picture the states evolve in time while the operator-observables are held fixed.

Example: Free particle

To give you an idea of how the Heisenberg picture looks. Let us briefly consider a free particle moving in 1-D. The Hamiltonian (in the Schrödinger picture) is

$$H = \frac{P^2}{2m}.$$

This is also the Hamiltonian in the Heisenberg picture because (exercise)

$$e^{\frac{i}{\hbar}(t-t_0)H} H e^{\frac{i}{\hbar}(t-t_0)H} = H.$$

This means that the energy eigenvectors are the same for all time. This means the probability distribution for energy is time independent, which we already knew.

Because P commutes with H and hence $U(t, t_0)$ (exercise) it is easy to see that

$$P(t) = e^{\frac{i}{\hbar}(t-t_0)H} P e^{\frac{i}{\hbar}(t-t_0)H} = P.$$

Thus we get to use the same operator for momentum (namely, the generator of translations) for all time *for a free particle*. This means that the momentum eigenvectors are the same for all time. This means the momentum probability distribution is time-independent, which we could already see in the Schrödinger picture.

The position does not commute with the Hamiltonian. With a little work you can show

$$e^{\frac{i}{\hbar}(t-t_0)H} X e^{\frac{i}{\hbar}(t-t_0)H} = X + \frac{t - t_0}{m} P.$$

I will show you how to derive this result later. Thus the operator representing the position observable at time t is a combination of the Schrödinger position and momentum operators. This means the position eigenvectors change in time. This means the position probability distribution changes in time, as it should for a particle which is supposed to be moving. Note that both the position and momentum operators in the Heisenberg picture are related to the initial (Schrödinger) operators via the solutions of the classical equations of motion. This is not an accident and will be explained shortly.