Lecture 13

Relevant sections in text: §2.1

Stationary states

Suppose the Hamiltonian does not depend upon time. If the initial state vector for a system is an energy eigenvector,

$$|\psi, t_0\rangle = |E_k\rangle,$$

(so that the energy is known with certainty to be E_k at $t = t_0$) then the state vector changes in a trivial fashion as time progresses. This can be seen in 2 ways. Firstly, simply apply the time evolution operator:

$$U(t, t_0)|\psi, t_0\rangle = e^{-\frac{i}{\hbar}(t-t_0)H}|\psi, t_0\rangle = e^{-\frac{i}{\hbar}(t-t_0)E_k}|\psi, t_0\rangle.$$

The state vector at time t is just a phase time the state at time t_0. Such state vectors are physically equivalent since probability distributions (*i.e.,* expectation values) are not changed by a phase transformation. You can also check this result by using the 3 step process described above (exercise). The energy eigenvectors change only by a phase factor, hence *all physical predictions (probability distributions) will not change in time.* Thus energy eigenvectors are states in which the system exhibits "stationary" behavior. Such states are naturally called *stationary states.* Nothing ever happens in a stationary state.

Often times we prepare the initial state by a filtering process based upon measurement of one (or more commuting) observables. If these observables commute with the Hamiltonian, then one can arrange that the prepared/filtered states are energy eigenvectors, thus stationary states.

Let me point out a paradox here. Surely you have heard of the process called "spontaneous emission". You know, where you put an atomic electron in an excited state and even if you do nothing to the atom it will decay to a lower energy state? And I know you have learned about energy levels of atoms in a previous course in quantum mechanics. The energy levels are eigenvectors of the Hamiltonian for the atom, *i.e.,* stationary states. Now the paradox appears: how can an excited state of the atom — a stationary state — decay into anything if nothing every happens in a staionary state? Think about it!

Conservation of energy

The state vector at time t can be expressed as

$$|\psi, t\rangle = \sum_j c_j e^{-\frac{i}{\hbar}(t-t_0)E_j}|E_j\rangle,$$

where

$$|\psi, t_0\rangle = \sum_j c_j |E_j\rangle.$$

The components of a state vector along the energy eigenvector basis determine the probability distribution for the energy in that state at any given time t. Time evolution amounts to multiplying these coefficients by phase factors. In general, this changes the vector in a non-trivial way; this causes probability distributions for various observables to change in time. However, since the probabilities for energy arise via the absolute values of the components, you can easily see that the probability distribution for energy does *not* change in time (still assuming that the Hamiltonian is time independent). For example, assuming E_k is non-degenerate, we have

$$Prob(H = E_k, t) = |c_k e^{-\frac{i}{\hbar}(t-t_0)E_k}|^2 = |c_k|^2.$$

(If E_k is degenerate we sum the probabilities over a basis of states with energy E_k.) Thus the probability distribution for energy does not change in time. This is how energy is conserved in quantum mechanics. More generally, any conserved quantity is characterized by a probability distribution which is time independent. We will discuss this shortly.

Of course, in classical mechanics we usually consider conservation of energy to be characterized by (i) measure the energy initially, (ii) measure the energy at some later time, (iii) energy is conserved if these results are the same. One can do that here as well, but if you measure the energy initially you will then have prepared/filtered your system state to be an energy eigenvector. Then, of course, the probability distribution is quite simple! As we have seen, if the system is in an energy eigenvector at one time, it remains there for all time. Thus you will always get the same result for any subsequent measurement of energy. That is certainly much like the classical way of viewing conservation of energy. However, as we have seen, in a stationary state *all* probability distributions are time independent. In non-stationary states, where the dynamical evolution is non-trivial, one cannot say with statistical certainty what the initial energy is – one has a non-trivial probability distribution. What one *can* say is that, while other probability distributions will, in general, change in time (energy-compatible observables being the exceptions), the energy probability distribution will not. This is what conservation of energy means in quantum mechanics.

Example: Spin 1/2 in a uniform magnetic field.

Let us consider the dynamical evolution of an electron's spin in a (uniform) magnetic field. We ignore the translational degrees of freedom of the electron. The electron magnetic moment is an observable which we represent as

$$\vec{\mu} = -\left(\frac{e}{mc}\right)\vec{S},$$

(Here $e > 0$ is the magnitude of the electron electric charge.) The Hamiltonian for a magnetic moment in a (uniform, static) magnetic field $\vec{B}$ is taken to be

$$H = -\vec{\mu} \cdot \vec{B} = \left(\frac{e}{mc}\right) \vec{S} \cdot \vec{B}.$$

Let us choose the z axis along $\vec{B}$ so that

$$H = \left(\frac{eB}{mc}\right) S_z.$$

Evidently, S_z eigenvectors are energy eigenvectors. Thus the S_z eigenvectors are stationary states; in such states the probability distributions for all observables, *i.e.,* $\vec{S}$, energy, *etc.* are time independent.

Let us consider the time evolution of a general initial state, which we can expand in the energy/S_z basis:

$$|\psi, t_0 = 0\rangle = a|+\rangle + b|-\rangle, \quad |a|^2 + |b|^2 = 1.$$

Note that $|a|^2$ is both the probability for getting $S_z = \hbar/2$ and for getting energy $\frac{eB\hbar}{2mc}$ — similar comment for $|b|^2$. These probabilities are time independent. We have

$$\begin{aligned}|\psi, t\rangle &= e^{-\frac{i}{\hbar}\frac{eB}{mc}tS_z}|\psi, 0\rangle \\ &= ae^{-i\omega t/2}|+\rangle + be^{i\omega t/2}|-\rangle,\end{aligned}$$

where

$$\omega = \frac{eB}{mc},$$

is the classical precession frequency of a magnetic moment in a magnetic field. Thus the effect of the magnetic field on the state vector is to cause its components in the energy basis to oscillate sinusoidally with a frequency of $\omega/2$. Note that the components actually oscillate with 1/2 the classical precession frequency. From this formula you can also see that if the initial state is an S_z/energy eigenvector, then it remains so for all time. To see dynamical evolution, we pick an initial state that is not an energy eigenvector. For example, suppose that at $t = 0$, $a = b = \frac{1}{\sqrt{2}}$, *i.e.,* $|\psi, 0\rangle = |S_x, +\rangle$. What is the probability for getting $S_x = \pm\frac{\hbar}{2}$ at time t? We have (exercise)

$$Prob(S_x = \frac{\hbar}{2}, t) = |\langle S_x, +|\psi(t)\rangle|^2 = \cos^2(\frac{\omega t}{2}),$$

$$Prob(S_x = -\frac{\hbar}{2}, t) = |\langle S_x, -|\psi(t)\rangle|^2 = \sin^2(\frac{\omega t}{2}).$$

Thus an effect of the magnetic field is to cause the x-component of the spin to periodically "flip" – but the frequency of the flip is 1/2 the classical frequency. One can also visualize the behavior of the spin by following its expectation value in time. Still using $|\psi, 0\rangle = |S_x, +\rangle$ we have (good exercise)

$$\langle S_x\rangle(t) = \langle\psi, t|S_x|\psi, t\rangle = \frac{\hbar}{2}\cos\omega t, \quad \langle S_y\rangle(t) = \langle\psi, t|S_x|\psi, t\rangle = \frac{\hbar}{2}\sin\omega t, \quad \langle S_z\rangle(t) = 0.$$

Thus, on the average, the spin vector precesses about the axis determined by the magnetic field, and in the plane orthogonal to it, with angular velocity given by the classical precession frequency $\omega = \frac{eB}{mc}$. The expectation values behave "classically", while the probability distributions do not.

Time rate of change of expectation values

Given the last result on the time evolution of expectation values, let us consider how expectation values change in time in a general setting. Assuming that the definition of the observable of interest does not depend upon time, so that its operator representative does not change in time in the Schrödinger picture – which we are still using – we have

$$\frac{d}{dt}\langle A\rangle(t) = \frac{d}{dt}\langle\psi, t|A|\psi, t\rangle.$$

Using

$$\frac{d}{dt}|\psi, t\rangle = \frac{1}{i\hbar}H|\psi, t\rangle, \quad \frac{d}{dt}\langle\psi, t| = -\frac{1}{i\hbar}\langle\psi, t|H,$$

we get (exercise)

$$\frac{d}{dt}\langle A\rangle(t) = \frac{1}{i\hbar}\langle[A, H]\rangle(t).$$

Thus, dynamical evolution of the expectation value of an observable is controlled by its incompatibility with H. If A and H are compatible then the expectation value is time-independent. In fact, in this case H and A admit a common basis of eigenvectors whence the probability distribution for A is time independent for the same reason it is for H (as we mentioned previously — also, see below). As an exercise you can use the result shown above to derive the time rate of change of the expectation values that we studied in the spin 1/2 precession example in the last lecture.

Conservation laws

We have already seen (assuming $\frac{\partial H}{\partial t} = 0$) that H (energy) is conserved in the sense that the probability distribution for H is time-independent. In a special class of states, the stationary states, the energy is known with certainty for all time and *all* other probability distributions are time independent. It is possible to have conserved quantities besides the energy of course without choosing a stationary state. We say that an observable A is conserved if its probability distribution is always time independent – irrespective of the initial state of the system. It is not hard to see that A is conserved if and only if it is compatible with H:

$$[A, H] = 0.$$

The "if" part of the statement is proved as follows. If the commutator vanishes, then the basis of energy eigenvectors can be chosen to also be A eigenvectors. Let us denote those eigenvectors by $|k\rangle$, where

$$A|k\rangle = a_k|k\rangle, \quad H|k\rangle = E_k|k\rangle.$$

As before, the initial state can be expanded as

$$|\psi, t_0\rangle = \sum_k c_k|k\rangle,$$

where $|c_k|^2$ is the probability for getting a_k and/or E_k upon measurement of the observable (represented by) A and/or H. Because the basis is built from H eigenvectors we still have

$$|\psi, t\rangle = \sum_k c_k e^{-\frac{i}{\hbar}E_k(t-t_0)}|k\rangle,$$

where

$$|c_k e^{-\frac{i}{\hbar}E_k(t-t_0)}|^2 = |c_k|^2$$

is the probability for getting a_k at time t. Thus the probability distribution for A is time-independent.

The "only if" part of the statement is proved as follows. Note that if the probability distribution for A is to be time-independent then, using the results of the previous section, we must have

$$\langle\psi, t_0|[A, H]|\psi, t_0\rangle = 0$$

for all possible initial state vectors $|\psi, t_0\rangle$. As we have mentioned before, if an operator on a complex Hilbert space has vanishing diagonal matrix elements then it is the zero operator.