Lecture 10

Relevant sections in text: §1.7

Relation between position and momentum wave functions

A very useful and important relationship exists between the position and momentum (generalized) eigenvectors. To get at it, we study the scalar product $\langle x|p\rangle$, which can be viewed as the position wave function representing a momentum eigenvector. (Similarly, $\langle p|x\rangle = \langle x|p\rangle^*$ can also be viewed as the the momentum wave function representing a position eigenvector.) This complex function of x must satisfy (for each p)

$$\langle x-\epsilon|p\rangle = \langle x|T_\epsilon|p\rangle = e^{-\frac{i}{\hbar}\epsilon p}\langle x|p\rangle.$$

This implies (to first order in ϵ) (exercise)

$$\frac{d}{dx}\langle x|p\rangle = \frac{i}{\hbar}p\langle x|p\rangle.$$

The solution to this equation is

$$\langle x|p\rangle = (const.)e^{\frac{i}{\hbar}px}.$$

The constant can be determined by the normalization condition:

$$\delta(p,p') = \langle p|p'\rangle = \int_{-\infty}^{\infty} dx\langle p|x\rangle\langle x|p'\rangle.$$

Using the Fourier representation of the delta function,

$$\delta(p,p') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ix(p'-p)},$$

we see that (exercise)

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}.$$

Thus we have recovered a familiar result from wave mechanics: the position space wave function for a particle in a (idealized) state such that the momentum has the value p is a (complex) plane wave* with wavelength $\frac{2\pi\hbar}{p}$. Because the absolute value of the wave function is unity, the particle has an equal probability of being found anywhere (think: uncertainty relation). Note also that since the energy of a free particle of mass m is

$$H = \frac{P^2}{2m},$$

* Of course, plane waves are not normalizable, but we have already discussed this subtlety.

this wave function describes a free particle with energy $p^2/2m$.

Because

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px},$$

we see that the momentum space probability amplitude for a particle in an idealized state corrsponding to the particle having a definite spatial location is also a plane wave. We see that for an arbitrarily well-localized particle, all momenta are equally likely (uncertainty relation again).

With these results in hand we can give an explicit relation between the position and momentum bases:

$$|x\rangle = \int_{-\infty}^{\infty} dp|p\rangle\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\, e^{-\frac{i}{\hbar}px}|p\rangle,$$

and

$$|p\rangle = \int_{-\infty}^{\infty} dx|x\rangle\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{\frac{i}{\hbar}px}|x\rangle.$$

If we set

$$\psi(x) = \langle x|\psi\rangle, \quad \tilde{\psi}(p) = \langle p|\psi\rangle,$$

we get (exercise)

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\, e^{\frac{i}{\hbar}px}\tilde{\psi}(p),$$

and

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-\frac{i}{\hbar}px}\psi(x).$$

Thus we recover the standard result that the position wave functions and momentum wave functions are related by Fourier transforms. Note also that (exercise)

$$\langle\psi|\phi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\phi(x) = \int_{-\infty}^{\infty} dp\, \tilde{\psi}^*(p)\tilde{\phi}(p).$$

When representing states by position (momentum) wave functions we say we are using the position (momentum) representation for the quantum system. In the momentum representation the momentum operator is a "multiplication operator":

$$P\tilde{\psi}(p) = \langle p|P|\psi\rangle = p\langle p|\psi\rangle = p\tilde{\psi}(p),$$

while the position operator is a "differentiation operator":

$$\begin{aligned}
X\tilde{\psi}(p) &= X\left\{\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-\frac{i}{\hbar}px}\psi(x)\right\} \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-\frac{i}{\hbar}px}x\psi(x) \\
&= -\frac{\hbar}{i}\frac{\partial}{\partial p}\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-\frac{i}{\hbar}px}\psi(x) \\
&= -\frac{\hbar}{i}\frac{\partial\tilde{\psi}(p)}{\partial p}.
\end{aligned}$$

Expectation values

Expectation values in the position and momentum representation are easy to compute. Using the notation

$$\psi(x) = \langle x|\psi\rangle, \quad \tilde{\psi}(p) = \langle p|\psi\rangle,$$

we have, in particular, the following

$$\langle\psi|f(X)|\psi\rangle = \int_{-\infty}^{\infty} dx f(x)|\psi(x)|^2 = \int_{-\infty}^{\infty} dp\, \tilde{\psi}^*(p) f(-\frac{\hbar}{i}\frac{d}{dp})\tilde{\psi}(p),$$

and

$$\langle\psi|f(P)|\psi\rangle = \int_{-\infty}^{\infty} dp f(p)|\tilde{\psi}(p)|^2 = \int_{-\infty}^{\infty} dx\, \psi^*(x) f(\frac{\hbar}{i}\frac{d}{dx})\psi(x),$$

You should prove these as a very good exercise.

Particle in 3 dimensions

The generalization to a particle in 3 dimensions is done by, essentially, tripling our previous constructions. Here we briefly describe this generalization.

Our strategy is to take account of the fact that we can measure positions in 3-d, and we can measure momenta in 3-d. Thus we now have position vectors $\mathbf{X}$ corresponding to 3 compatible observables: $X^i = (X, Y, Z)$, and momentum vectors $\mathbf{P}$ corresponding to 3 compatible observables: $P_i = (P_x, P_y, P_z)$. Each pair (X^i, P_i) is represented by self-adjoint operators exactly as before. The operator P_i is to generate translations in the corresponding position variable X^i. We therefore demand that they have the following *canonical commutation relations*:

$$[X^i, X^j] = [P_i, P_j] = 0, \quad [X^i, P_j] = i\hbar\delta^i_j I.$$

We have position (generalized) eigenvectors $|\mathbf{x}\rangle$ and momentum (generalized) eigenvectors $|\mathbf{p}\rangle$,

$$X^i|\mathbf{x}\rangle = x^i|\mathbf{x}\rangle, \quad P_i|\mathbf{p}\rangle = p_i|\mathbf{p}\rangle.$$

These form a (generalized) basis:

$$\int d^3x\, |\mathbf{x}\rangle\langle\mathbf{x}| = I = \int d^3p\, |\mathbf{p}\rangle\langle\mathbf{p}|.$$

Here it is understood that the integrals run over all of position/momentum space.

The self-adjoint momentum operators generate unitary transformations corresponding to 3-d translations:

$$T_{\mathbf{a}} = e^{-\frac{i}{\hbar}\mathbf{a}\cdot\mathbf{P}}, \quad T_{\mathbf{a}}|\mathbf{x}\rangle = |\mathbf{x}+\mathbf{a}\rangle.$$

The canonical commutation relations reflect the fact that translations are commutative operations:

$$T_{\mathbf{a}}T_{\mathbf{b}} = T_{\mathbf{b}}T_{\mathbf{a}} = e^{-\frac{i}{\hbar}(\mathbf{a}+\mathbf{b})\cdot\mathbf{P}} = T_{\mathbf{a}+\mathbf{b}}.$$

Note that the commutation relations allow us to choose a basis of simultaneous (generalized) eigenvectors of the position or momentum. The relation between the two bases is

$$\langle\mathbf{x}|\mathbf{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}.$$

The position wave functions and momentum wave functions are defined as usual by taking the components of a state vector along the corresponding basis:

$$\psi(\mathbf{x}) = \langle\mathbf{x}|\psi\rangle, \quad \tilde{\psi}(\mathbf{p}) = \langle\mathbf{p}|\psi\rangle.$$

We then have

$$X^i\psi(\mathbf{x}) = x^i\psi(\mathbf{x}), \quad P_i\psi(\mathbf{x}) = \frac{\hbar}{i}\frac{\partial}{\partial x^i}\psi(\mathbf{x}),$$

and

$$X^i\tilde{\psi}(\mathbf{p}) = -\frac{\hbar}{i}\frac{\partial}{\partial p_i}\tilde{\psi}(\mathbf{p}), \quad P_i\tilde{\psi}(\mathbf{p}) = p_i\tilde{\psi}(\mathbf{p}),$$

The probability distributions for finding position/momentum in a volume $V/\tilde{V}$ are

$$Prob(\mathbf{X}\in V) = \int_V d^3x\,|\psi(\mathbf{x})|^2, \quad Prob(\mathbf{P}\in\tilde{V}) = \int_{\tilde{V}} d^3p\,|\tilde{\psi}(\mathbf{p})|^2.$$