Lecture 9

Relevant sections in text: §1.6, 1.7

Momentum (cont.)

The translation operator, is defined by

$$T_a|x\rangle = |x+a\rangle.$$

On an arbitrary vector it acts as

$$T_a|\psi\rangle = T_a \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x+a\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x-a|\psi\rangle.$$

T_a maps normalized states into normalized states:

$$1 = \langle\psi|\psi\rangle = \langle\psi|T_a^\dagger T_a|\psi\rangle.$$

It can be shown that this property (for all vectors) means

$$T_a^\dagger T_a = T_a T_a^\dagger = I, \quad \Longleftrightarrow \quad T_a^\dagger = T_a^{-1}.$$

We say that the operator T satisfying this last set of relations is *unitary*. Note that a unitary operator preserves all scalar products (exercise).

Note that for position wave functions we have

$$T_a\psi(x) = \langle x|T_a|\psi\rangle = \langle x-a|\psi\rangle = \psi(x-a).$$

So, moving the system to the right by an amount a shifts the argument of the wave function to the *left*. (To see that this makes sense, consider for example a particle with a Gaussian wave function (exercise).) In terms of wave functions, the unitarity of T_a is expressed as

$$\int_{-\infty}^{\infty} |\psi(x-a)|^2 = \int_{-\infty}^{\infty} |\psi(x)|^2 = 1.$$

As we mentioned above, momentum is identified with the infinitesimal generator of translations. Thus, consider an infinitesimal translation, T_ϵ, $\epsilon << 1$. We assume that T_a is continuous in a so that we may expand the operator in a Taylor series

$$T_\epsilon = I - \epsilon\frac{i}{\hbar}P + \mathcal{O}(\epsilon^2).$$

The mathematical definition of the Taylor series of an operator needs a fair amount of discussion, which we will suppress. For our purposes you can just interpret the expansion as meaning that any matrix elements of the operator can be so expanded. The factor of $-\frac{i}{\hbar}$ has been inserted for later convenience. Here P is a linear operator, called the *infinitesimal generator of translations.** The unitarity and continuity of T implies that P is self-adjoint. Indeed, considering the $\mathcal{O}(\epsilon)$ terms you can easily see (exercise)

$$I = T_\epsilon^\dagger T_\epsilon = (I + \epsilon\frac{i}{\hbar}P^\dagger + \mathcal{O}(\epsilon^2))(I - \epsilon\frac{i}{\hbar}P + \mathcal{O}(\epsilon^2)) \Longrightarrow P = P^\dagger.$$

In fact, the self-adjointness of P is also sufficient for T to be unitary. This can be seen by representing T_a as an infinite product of infinitesimal transformations:

$$T_a = \lim_{N\to\infty}(I - \frac{i}{\hbar}\frac{a}{N}P)^N = e^{-\frac{i}{\hbar}aP}.$$

It is not hard to check that any operator of the form e^{iA} with $A^\dagger = A$ is unitary (exercise). Thus P represents an observable, which we identify with the momentum of the particle.

The canonical commutation relations

We now consider the commutation relation between position and momentum. We can derive this relation by studying the commutator between position and translation operators and then considering the limit in which the translation is infinitesimal. Check the following computations as an exercise.

$$XT_\epsilon|x\rangle = (x+\epsilon)|x+\epsilon\rangle.$$

$$T_\epsilon X|x\rangle = x|x+\epsilon\rangle.$$

Subtracting these two relations, taking account of the definition of momentum, and working consistently to first-order in ϵ we have (exercise)

$$X(-\frac{i}{\hbar}P)|x\rangle - (-\frac{i}{\hbar}P)X|x\rangle = |x\rangle.$$

This implies – keep in mind that $|x\rangle$ is a (generalized) basis –

$$[X, P] = i\hbar I.$$

This relation, along with the (trivial) relations

$$[X, X] = 0 = [P, P],$$

constitute the *canonical commutation relations* for a particle moving in one dimension.

* Technically, the infinitesimal generator is $-\frac{i}{\hbar}P$, but it is a convenient abuse of terminology – and it is customary – to call P the infinitesimal generator.

Note that these commutation relations show us that position and momentum are *incompatible* observables. They thus satisfy an uncertainty relation:

$$\langle \Delta X^2 \rangle \langle \Delta P^2 \rangle \geq \frac{1}{4}\hbar^2.$$

This is the celebrated position-momentum uncertainty relation. It shows that if you try to construct a state with a very small dispersion in X (or P) then the dispersion in P (or X) must become large. Note also that the uncertainty relation shows the dispersion in position or and/or momentum can never vanish.⋆ However, either of them can be made arbitrarily small provided the other observable has a sufficiently large dispersion.

Momentum as a derivative

It is now easy to see how the traditional representation arises in which momentum is a derivative operator acting on position wave functions. Consider the change in a position wave function under an infinitesimal translation. We have

$$\psi(x-\epsilon) = T_\epsilon \psi(x) = (I - \frac{i}{\hbar}P + \mathcal{O}(\epsilon^2))\psi(x),$$

and we have

$$\psi(x-\epsilon) = \psi(x) - \epsilon\frac{d\psi(x)}{dx} + \mathcal{O}(\epsilon^2).$$

Comparing terms of order ϵ we see that

$$P\psi(x) \equiv \langle x|P|\psi\rangle = \frac{\hbar}{i}\frac{d\psi(x)}{dx}.$$

Of course, you can now easily verify the position wave function representation of the canonical commutation relations:

$$[X, P]\psi(x) = i\hbar\psi(x).$$

Momentum wave functions

We have already indicated that one can use any continuous observable to define a class of wave functions. We have used the position observable to this end. Now let us consider using the momentum observable. We define, as usual,*

$$P|p\rangle = p|p\rangle, \quad p \in \mathbf{R}$$

⋆ This underscores the fact that, strictly speaking, position and momentum "eigenvectors" cannot be defined in the Hilbert space.

* Of course, these properties, *e.g.,* the spectrum of P, ought to be derived directly from the definition of the translation operator. This can be done, but we won't do it explicitly here.

$$I = \int_{-\infty}^{\infty} dp|p\rangle\langle p|,$$

$$\psi(p) = \langle p|\psi\rangle, \quad P\psi(p) = p\psi(p),$$

and so forth. The interpretation of the momentum wave function is that $|\psi(p)|^2\,dp$ is the probability to find the momentum in the range $[p, p+dp]$. In other words,

$$Prob(P \in [a,b]) = \int_a^b dp\,|\psi(p)|^2.$$

Note that a translation of a momentum wave function is a simple phase transformation (exercise):

$$T_a\psi(p) = \langle p|e^{-\frac{i}{\hbar}aP}|\psi\rangle = e^{-ipa}\psi(p).$$

Physically, this means that a translation has no effect on the momentum probability distribution (exercise). Exercise: use the momentum wave function representation to check the unitarity of T_a.