Lecture 5

Relevant sections in text: §1.4

Probability interpretation

We are now ready to use our definition of state vectors and operator-observables to extract physical predictions. We use the postulate relating expectation values to diagonal matrix elements. Let us start with the eigenvector $|+\rangle$ of S_z. *In the state represented by this vector* we easily get

$$\langle S_z\rangle = \langle +|S_z|+\rangle = \frac{\hbar}{2},$$

With just a little more work we get

$$\langle S_x\rangle = \langle S_y\rangle = 0.$$

For example:

$$S_x = \frac{\hbar}{2}\Big((|+\rangle\langle -| + |-\rangle\langle +|\Big),$$

so that

$$\langle S_x\rangle = \langle +|S_x|+\rangle = \frac{\hbar}{2}\langle +|\Big((|+\rangle\langle -| + |-\rangle\langle +|\Big)|+\rangle = 0.$$

These results match the results we found in the Stern-Gerlach experiment. Indeed, $|+\rangle \equiv |S_z,+\rangle$ is supposed to be a state where S_z is known with certainty (probability unity) to have the value $+\hbar/2$. Thus the expectation value of this observable in such a state is also $\hbar/2$. We also saw in the Stern-Gerlach experiment that this state has equal probability of finding $\pm\hbar/2$ for S_x and S_y whence their expectation value vanishes, in agreement with the output of the mathematical model. You can verify as an exercise that similar comments (with appropriate permutation of x, y, and z) can be made for $|S_z,-\rangle$, $|S_x,\pm\rangle$, *etc.*

The postulate of QM defining expectation values is the place where the mathematical representation of a physical system makes contact with reality. It provides the predictions that can be tested/compared with experiment. Note that this postulate gives the physical output of QM in terms of probabilities (specifically, expectation values). In fact, as we shall see, *all* the physical predictions of quantum mechanics are intrinsically probabilistic.

How can we see directly the probabilities for the various outcomes of a measurement of something like spin when the expectation value postulate only gives statistical averages, *i.e.,* expectation values? We proceed as follows. Consider a function $f(x)$ that takes the value 1 at, say, $x = \hbar/2$ and vanishes otherwise.* Consider the observable, say, $f(S_x)$ –

* Such a function is called a *characteristic function* for the set $x = \frac{\hbar}{2}$.

not yet viewed as an operator, but just as the experimentally accessible observable. So, $f(S_x)$ is a mathematical model for a detector for spin-up along the x-axis — it yields the value one when the spin along x is $\frac{\hbar}{2}$ and yields zero otherwise. If you think about repeatedly setting up an experimental state and monitoring the state of the "detector" $f(S_x)$ you will see that (in the limit of an arbitrarily large number of experiments) the average value of $f(S_x)$ over many experimental runs – the expectation value of $f(S_x)$ – is precisely equal to the probability for finding S_x to have the value $\hbar/2$ (exercise). More generally, the probability of finding S_x to have a value in any range R of real numbers is just the expectation value of the characteristic function of the set R. Clearly we can do the same for any other component of the spin. Thus we can extract a probability for a given outcome of a measurement of an observable by computing an expectation value of a suitable (characteristic) function of the observable. Apparently to implement this idea we need to figure out how to define functions of operators. Let's see how to do this.

In general, given a Hermitian operator A with an ON basis of eigenvectors $|i\rangle$ and eigenvalues a_i, and a real-valued function $h(x)$, we want to define a Hermitian operator $h(A)$. If you think a moment about simple functions like polynomials, you will see that this definition should be such that the eigenvectors of $h(A)$ will be the same as for A, and the eigenvalues will be $h(a_i)$. Evidently we desire the spectral decomposition

$$h(A) = \sum_i h(a_i)|i\rangle\langle i|.$$

You can easily check that with this definition $|i\rangle$ do indeed constitute (a basis of) eigenvectors of $h(A)$ with eigenvalues $h(a_i)$. Thus we define functions of observables by their spectral decomposition.

Returning to probabilities for spin measurements, given the characteristic function $f(x)$ for $x = \hbar/2$, we *define* the *operator* $f(S_x)$ by its spectral decomposition:

$$f(S_x) = f(\frac{\hbar}{2})|S_x,+\rangle\langle S_x,+| + f(-\frac{\hbar}{2})|S_x,-\rangle\langle S_x,-| = |S_x,+\rangle\langle S_x,+|.$$

The desired operator is just the projection operator into the desired eigenspace.* It is now easy to see, by computing expectation values of $f(S_x)$ according to the expectation value postulate, that the following probability distributions arise (good exercise!):

State: $|S_z,\pm\rangle \longrightarrow$ *Prob*$(S_x = \pm\hbar/2) = 1/2, \quad$ *Prob*$(S_x = \mp\hbar/2) = 1/2$

State: $|S_x,\pm\rangle \longrightarrow$ *Prob*$(S_x = \pm\hbar/2) = 1, \quad$ *Prob*$(S_x = \mp\hbar/2) = 0$

* You can easily see from the previous paragraph that this is a general result, not limited to the spin 1/2 example.

State: $|S_y,\pm\rangle \longrightarrow Prob(S_x = \pm\hbar/2) = 1/2, \quad Prob(S_x = \mp\hbar/2) = 1/2.$

To check these formulas you need to use the easily verified results:

$$\langle S_x,\pm|+\rangle = \frac{1}{\sqrt{2}} \quad \langle S_x,\pm|-\rangle = \pm\frac{1}{\sqrt{2}}.$$

You can easily play similar games with other components of $\vec{S}$. You can also compute the probabilities (via expectation values) in any state you like just by expanding the state in the $|\pm\rangle$ basis and computing the expectation values using the orthonormality of the basis. In particular if a state (unit) vector takes the form

$$|\psi\rangle = a|+\rangle + b|-\rangle,$$

where the coefficients a and b are restricted by

$$1 = \langle\psi|\psi\rangle = |a|^2 + |b|^2,$$

then the probability of getting $\hbar/2$ for S_z is given by $|a|^2$ while the probability for getting $-\hbar/2$ is given by $|b|^2$. To see this we compute:

$$Prob(S_z = \pm\frac{\hbar}{2}) = \langle\psi|\pm\rangle\langle\pm|\psi\rangle = |\langle\pm|\psi\rangle|^2.$$

Note that the normalization condition

$$1 = \langle\psi|\psi\rangle = |a|^2 + |b|^2$$

guarantees that these probabilities add up to unity. This implies that the probability for getting any other values for S_z must vanish. Let us prove this directly. Let $g(x)$ be a function that vanishes at $\pm\hbar/2$, that is, that vanishes at the eigenvalues of any of the spin operators. For any component of the spin, S_k, and for any state $|\psi\rangle$ we have that

$$\langle g(S_k)\rangle = \langle\psi|\left\{g(\frac{\hbar}{2})|S_k,+\rangle\langle S_k,+| + g(-\frac{\hbar}{2})|S_k,-\rangle\langle S_k,-|\right\}|\psi\rangle = 0.$$

In particular, if you pick g to be a characteristic function of any set not including the spectrum of S_k, then the expectation value – *which is the probability for finding S_k to be in that set* – vanishes. Thus we see that *the only possible outcome of a measurement of an observable is an element of its spectrum, i.e., one of its eigenvalues.*

Here is a handy observation. Suppose we have a normalized state $|\psi\rangle$ and we want to know the probability of finding, say, $S_x = \frac{\hbar}{2}$. How do we compute it? In principle we can expand the state in terms of the basis $|S_x,\pm\rangle$:

$$|\psi\rangle = c|S_x,+\rangle + d|S_x,-\rangle, \quad |c|^2 + |d|^2 = 1.$$

The probability is then given by $|c|^2$, using the exact same logic as before. But here's a shortcut: Given a state vector $|\psi\rangle$, and given any (normalized) eigenvector $|\lambda\rangle$ of some observable corresponding to a non-degenerate eigenvalue λ, the component of $|\psi\rangle$ along $|\lambda\rangle$ is just $\langle\lambda|\psi\rangle$, whence the probability for getting λ is just $|\langle\lambda|\Psi\rangle|^2$. So, in our spin example, given the state vector $|\psi\rangle$, the probability for getting $\pm\hbar/2$ upon measurement of $\hat{n}\cdot\vec{S}$ is $|\langle\hat{n}\cdot\vec{S},\pm|\psi\rangle|^2$.

The foregoing results are Very Important and are easily generalized to other quantum systems; they are at the heart of the physical output of quantum mechanics. Let us summarize what we found.

(i) The only possible outcome of a measurement of a spin component is one of the eigenvalues of the corresponding operator ($\pm\hbar/2$).

(ii) Given a state represented by the (unit) vector $|\psi\rangle$ and given an observable (represented by) $\hat{n}\cdot\vec{S}$, the probability for getting the value $\pm\hbar/2$ upon measurement of (the observable represented by) $\hat{n}\cdot\vec{S}$ is $|\langle\hat{n}\cdot\vec{S},\pm|\psi\rangle|^2$. Note that the normalization of the state vector guarantees that the probabilities for all possible outcomes of a measurement add up to unity.