Lecture 4

Relevant sections in text: §1.2, 1.3

Linear operators (cont.)

Given a linear operator A we recall the notion of eigenvectors and eigenvalues. They are solutions λ and $|\lambda\rangle$ to the equation

$$A|\lambda\rangle = \lambda|\lambda\rangle.$$

The zero vector is not considered an eigenvector. Note that the eigenvector $|\lambda\rangle$ corresponding to an eigenvalue λ is not unique since any scalar multiple of the eigenvector will also be an eigenvector with the same eigenvalue (exercise). However, note that such eigenvectors are not linearly dependent.* It may or may not happen that there is more than one linearly independent eigenvector for a given eigenvalue. When there is, one says that the eigenvalue is *repeated* or, more commonly, *degenerate*. As an extreme example, the identity operator has a degenerate eigenvalue (unity); the dimension of the set of eigenvectors with this eigenvalue is that of the whole Hilbert space. In general, the number of eigenvalues and the dimension of the eigenspace can range from 0 to the (currently finite) dimension of the Hilbert space.

Finally, let us note that a linear operator A on $\mathcal{H}$ also defines a linear operation on the dual vector space, *i.e.,* the space of bras. This operation is denoted

$$\langle\psi| \to \langle\psi|A.$$

To define $\langle\psi|A$ we should tell how it acts (as a linear function) on kets; the definition is

$$(\langle\psi|A)|\phi\rangle = \langle\psi|A|\phi\rangle.$$

As an exercise you should check that $\langle\psi|A$ so-defined is indeed a linear function, *i.e.,* a bra and that A is a linear operation on bras. Thus in a matrix element $\langle\phi|A|\psi\rangle$ you can view A as acting to the left or to the right!0

Self-adjoint operators

A linear operator A on a Hilbert space defines an operator $A^\dagger$ called the *adjoint* of A. It is defined by demanding that, for all kets, we have

$$\langle\psi|A^\dagger|\phi\rangle = \langle\phi|A|\psi\rangle^*.$$

* Recall that a set of vectors $|\alpha_i\rangle, i = 1, 2, \ldots, d$ is *linearly dependent* if there is a linear combination of them which vanishes, $\sum_i a_i|\alpha_i\rangle = 0$. If no such combination exists the set of vectors is called *linearly independent*. A basis is a maximal set of linearly independent vectors.

Note that this is equivalent to defining $A^\dagger$ as the operator whose matrix elements are the complex conjugate-transpose of those of A:

$$(A^\dagger)_{ij} = (A_{ji})^*.$$

Note that the bra corresponding to $A|\psi\rangle$ is

$$(A|\psi\rangle)^\dagger = \langle\psi|A^\dagger.$$

This follows directly from the definition of adjoint (exercise).

If

$$A^\dagger = A$$

we say that A is *self-adjoint* or *Hermitian.**

It is a standard result from linear algebra that a Hermitian operator has real eigenvalues and that eigenvectors corresponding to *distinct* eigenvalues must be orthogonal. See your text for the elementary proofs. An extremely important theorem from linear algebra says that a Hermitian operator always admits an orthonormal *basis* of eigenvectors. *This feature of Hermitian operators is absolutely critical for the physical interpretation of quantum mechanics.* For generic Hermitian operators we will typically use a notation such as

$$A|i\rangle = a_i|i\rangle, i = 1, 2, \ldots, d,$$

to denote the eigenvalues and corresponding basis of eigenvectors. Here d is the dimension of the vector space. This is not quite the same notation as used in your text, but I like my notation better and I can pretty much do what I want.

The spin operators

Finally we are ready to give the definition of the spin observables for a spin 1/2 system. We will do this by giving the expansion of the operators in projection operators built from a particular ON basis. We denote the basis vectors representing states in which the z component of spin is known by

$$|\pm\rangle := |S_z, \pm\rangle,$$

and define

$$S_x = \frac{\hbar}{2}\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)$$

$$S_y = i\frac{\hbar}{2}\Big(|-\rangle\langle+| - |+\rangle\langle-|\Big)$$

$$S_z = \frac{\hbar}{2}\Big(|+\rangle\langle+| - |-\rangle\langle-|\Big).$$

* For finite-dimensional vector spaces the two terms "Hermitian" and "self-adjoint" are equivalent. For infinite-dimensional spaces they are not quite the same. Although I shall not quibble much on this point, I will try to give a brief explanation of this at the appropriate time.

Note that we have picked a direction, called it z, and used the corresponding spin states for a basis. Of course, any other direction could be chosen as well. *You can now check that, with the above definition of S_z, $|\pm\rangle$ are in fact the eigenvectors of S_z with eigenvalues* $\pm\hbar/2$. Labeling matrix elements in this basis as

$$A_{ij} = \begin{pmatrix} \langle +|A|+\rangle & \langle +|A|-\rangle \\ \langle -|A|+\rangle & \langle -|A|-\rangle \end{pmatrix},$$

you can also verify the following matrix representations *in the $|\pm\rangle$ basis*:

$$(S_x)_{ij} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (S_y)_{ij} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (S_z)_{ij} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, you should check that all three spin operators are Hermitian. It will be quite a while before you get a deep understanding of why these particular operators are chosen. For now, let us just take them as given and see why they are appropriate to explain things like the Stern-Gerlach experiment.

As a good exercise you can verify that the eigenvectors of the other components of the spin operator are given by

$$|S_x, \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle), \quad |S_y, \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle).$$

The eigenvalues are $\pm\hbar/2$. Note that these eigenvectors are normalized to have norm ("length") unity. The fact that these eigenvectors are distinct from those of S_z will be dealt with a little later. For now, just note that the three operators do not share any eigenvectors. Note also that the eigenvalues of the spin operators are all non-degenerate – exercise.

Spectral decomposition

The spin operators have the general form

$$A = \sum_{ij} A_{ij}|i\rangle\langle j|,$$

which we discussed earlier. Note, though, that S_z has an especially simple, *diagonal* form. This is because it is being represented by an expansion in a basis of its eigenvectors. It is not hard to see that this result is quite general. If $|i\rangle$ is an ON basis of eigenvectors of A with eigenvalues a_i:

$$A|i\rangle = a_i|i\rangle,$$

then the matrix elements are

$$A_{ij} = \langle i|A|j\rangle = a_j\langle i|j\rangle = a_i\delta_{ij},$$

so that (exercise)

$$A = \sum_i a_i |i\rangle\langle i|.$$

This representation of an operator is called its *spectral decomposition.* The name comes from the terminology that the set of eigenvalues forms the *spectrum* of an operator (on a finite dimensional Hilbert space). You can easily see that the definition of S_z is its spectral decomposition.

Because every self-adjoint operator admits an ON basis of eigenvectors, each such operator admits a spectral decomposition. Of course, different operators will, in general, provide a different ON basis.