Spontaneous symmetry breaking. Goldstone's theorem. The Higgs mechanism.

Symmetry of laws versus symmetry of states.

Let us have a quick look at some of the classical field theoretic underpinnings of "spontaneous symmetry breaking" (SSB) in quantum field theory. Quite generally, SSB can be a very useful way of thinking about phase transitions in physics. In particle physics, SSB is used, in collaboration with the "Higgs mechanism", to give masses to gauge bosons (and other elementary particles) without destroying gauge invariance. We will explore some of this in due time. To begin to understand spontaneous symmetry breaking in field theory we need to refine our understanding of "symmetry", which is the goal of this section. The idea will be that there are two related kinds of symmetry one can consider: symmetry of the "laws" of motion governing the field, and symmetries of the "states" of the field.

So far we have been studying "symmetry" in terms of transformations of a (scalar) field which preserve the Lagrangian. For our present aims, it is good to think of this as a "symmetry of the laws of physics" in the following sense. The Lagrangian determines the "laws of motion" of the field via the Euler-Lagrange equations. As was pointed out in one of the problems, symmetries of a Lagrangian are also symmetries of the equations of motion. This means that if φ is a solution to the equations of motion and if $\tilde{\varphi}$ is obtained from φ via a symmetry transformation, then $\tilde{\varphi}$ also satisfies the *same* equations of motion. Just to be clear, let me cite a very elementary example. Consider the massless KG field described by the Lagrangian density:

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi. \tag{1}$$

It is easy to see that this Lagrangian admits the symmetry

$$\tilde{\varphi} = \varphi + const. \tag{2}$$

You can also easily see that the field equations

$$\partial^\alpha\partial_\alpha\varphi = 0 \tag{3}$$

admit this symmetry in the sense that if φ is solution then so is $\varphi + const$. Thus a symmetry of a Lagrangian is also a symmetry of the field equations and we will sometimes call it a symmetry of the law governing the field.

PROBLEM: It is not true that every symmetry of the field equations is a symmetry of the Lagrangian. Consider the massless KG field. Show that the scaling transformation $\tilde{\varphi} = (const.)\varphi$ is a symmetry of the field equations but is *not* a symmetry of the Lagrangian.

If the Lagrangian and its field equations are the "laws", then the solutions of the field equations are the "states" of the field that are allowed by the laws. The function $\varphi(x)$ is an allowed state of the field when it solves the field equations. A symmetry of a given "state", $\varphi_0(x)$ say, is then defined to be a transformation of the fields, $\varphi \to \tilde{\varphi}[\varphi]$, which preserves the given solution

$$\tilde{\varphi}[\varphi_0(x)] = \varphi_0(x). \tag{4}$$

Since symmetry transformations form a group, such solutions to the field equations are sometimes called "group-invariant solutions".

Let us consider an elementary example of group-invariant solutions. Consider the KG field with mass m. Use inertial Cartesian coordinates. We have seen that the spatial translations $x^i \to x^i + const.$ form a group of symmetries of the theory. It is easy to see that the corresponding group invariant solutions are of the form:

$$\varphi = A\cos(mt) + B\sin(mt), \tag{5}$$

where A and B are constants. Another very familiar type of example of group-invariant solutions you will have seen by now occurs whenever you are finding rotationally invariant solutions of PDEs.

PROBLEM: Derive the result (5).

An important result from the geometric theory of differential equations which relates symmetries of laws to symmetries of states goes as follows. Suppose G is a group of symmetries of a system of differential equations $\Delta = 0$ for fields φ on a manifold M, (*e.g.,* G is the Poincaré group). Let $K \subset G$ be a subgroup (*e.g.,* spatial rotations). Suppose we are looking for solutions to $\Delta = 0$ which are invariant under K. Then the field equations $\Delta = 0$ reduce to a system of differential equations $\hat{\Delta} = 0$ for K-invariant fields $\hat{\varphi}$ on the reduced space M/K.

As a simple and familiar example, consider the Laplace equation for functions on $\mathbf{R}^3$,

$$\partial_x^2\varphi + \partial_y^2\varphi + \partial_z^2\varphi = 0. \tag{6}$$

The Laplace equation is invariant under the whole Euclidean group G consisting of translations and rotations. Consider the subgroup K consisting of rotations. The rotationally invariant functions are of the form

$$\varphi(x, y, z) = f(r), \quad r = \sqrt{x^2 + y^2 + z^2}. \tag{7}$$

Rotationally invariant solutions to the Laplace equation are characterized by a reduced field f satisfying a reduced differential equation on the reduced space $\mathbf{R}^+$ given by

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{df}{dr}) = 0. \tag{8}$$

This is the principal reason one usually makes a "symmetry ansatz" for solutions to field equations which involves fields invariant under a subgroup K of the symmetry group G of the equations. It is not illegal to make other kinds of ansatzes, of course, but most will lead to inconsistent equations or equations with trivial solutions.

Having said all this, I should point out that just because you ask for group invariant solutions according to the above scheme it doesn't mean you will find any! There are two reasons for this. First of all the reduced differential equation may have no (or only trivial) solutions. Second it may be that there are no fields invariant under the symmetry group you are trying to impose on the state. As a simple example of this latter point, consider the symmetry group $\varphi \to \varphi + const.$ we mentioned earlier for the massless KG equation. You can easily see that there are no functions which are invariant under that transformation group. And I should mention that not all states have symmetry - indeed the generic states are completely asymmetric. States with symmetry are special, physically simpler states than what you expect generically.

To summarize, field theories may have two types of symmetry. There may be a group G of symmetries of its laws – the symmetry group of the Lagrangian (and field equations). There can be symmetries of states, that is, there may be subgroups of G which preserve certain states.

The "Mexican hat" potential

Let us now turn to a class of examples which serve to illustrate the preceding remarks and which we shall use to understand spontaneous symmetry breaking. We have actually seen these examples before.

We start by considering the real KG field with the double-well potential:

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi - (-\frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4). \tag{9}$$

As usual, we are working in Minkowski space with inertial Cartesian coordinates. This Lagrangian admits the Poincaré group as a symmetry group. It also admits the symmetry $\varphi \to -\varphi$, which forms a 2 element discrete subgroup Z_2 of the symmetry group of the Lagrangian. In an earlier homework problem we identified 3 simple solutions to the field equations for this Lagrangian:

$$\varphi = 0, \pm\frac{a}{b}, \tag{10}$$

where a and b are constants. These solutions are highly symmetric: they admit the whole Poincaré group of symmetries, as you can easily verify. Because Z_2 is a symmetry of the Lagrangian it must be a symmetry of the field equations, mapping solutions to solutions. You can verify that this is the case for the solutions (10). Thus the group consisting of the Poincaré group and Z_2 form a symmetry of the law governing the field. The 3 solutions in (10) represent 3 (of the infinite number of) possible solutions to the field equations – they are possible states of the field. The states represented by $\varphi = \pm\frac{a}{b}$ have Poincaré symmetry, but not Z_2 symmetry. In fact the Z_2 transformation maps between the solutions $\varphi = \pm\frac{a}{b}$. The state represented by $\varphi = 0$ has both the Poincaré and the Z_2 symmetry. The solution $\varphi = 0$ thus has more symmetry than the states $\varphi = \pm\frac{a}{b}$.

Let us consider the energetics of these highly symmetric solutions $\varphi = const$. In an inertial reference frame with coordinates $x^\alpha = (t, x^i)$, the conserved energy in a spatial volume V for this non-linear field is easily seen (from Noether's theorem) to be

$$E = \int_V dV \left(\frac{1}{2}\varphi_{,t}^2 + \frac{1}{2}\varphi_{,i}\varphi_{,}^{i} - \frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4\right). \tag{11}$$

You can easily check that the solutions given by $|\varphi| = 0, \frac{a}{b}$ are critical points of this energy functional. While it might be intuitively clear that these solutions ought to represent global minima at $\varphi = \pm\frac{a}{b}$ and a local maximum at $\varphi = 0$, it is perhaps not so easy to see this explicitly without some further analysis. We can investigate this as follows. Let us consider the change in the energy to quadratic-order in a displacement $u = u(t, x, y, z)$ from equilibrium in each case. We assume that u has compact support for simplicity. We write $\varphi = \varphi_0 + u$ where φ_0 is a constant and expand E to quadratic order in u. We get

$$E = -\frac{V}{4}\frac{a^4}{b^2} + \int_V dV \left\{\frac{1}{2}(u_{,t}^2 + u_{,i}u_{,}^{i}) + a^2u^2\right\} + \mathcal{O}(u^3), \quad \text{when } \varphi_0 = \pm\frac{a}{b} \tag{12}$$

and

$$E = \int_V dV \left\{\frac{1}{2}(u_{,t}^2 + u_{,i}u_{,}^{i}) - \frac{1}{2}a^2u^2\right\} + \mathcal{O}(u^3), \quad \text{when } \varphi_0 = 0. \tag{13}$$

Evidently, as we move away from $\varphi = \pm\frac{a}{b}$ the energy increases so that the critical points $\varphi = \pm\frac{a}{b}$ represent local minima. The situation near $\varphi = 0$ is less obvious. One thing is for sure: by choosing functions u which are suitably "slowly varying", one can ensure that the energy becomes negative in the vicinity of the solution $\varphi = 0$ so that $\varphi = 0$ is a saddle point if not a local maximum. We conclude that the state $\varphi = 0$ – the state of highest symmetry – is unstable and will not be seen "in the real world". On the other hand we expect the critical points $\varphi = \pm\frac{a}{b}$ to be stable. They are in fact the states of lowest energy and represent the possible ground states of the classical field. Evidently, the lowest energy is doubly degenerate.

Because the physical ground states have less symmetry than possible, one says that the symmetry group $Z_2 \times$ Poincaré has been "spontaneously broken" to just the Poincaré group for the ground state. This terminology is useful, but can be misleading. The theory retains the full symmetry group as a symmetry of its laws, of course. Compare this with, say, the ordinary Klein-Gordon theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi - m^2\varphi^2. \tag{14}$$

You can easily check that the solution $\varphi = 0$ is the global minimum energy state of the theory and that it admits the full symmetry group $Z_2 \times$ Poincaré. There is evidently no spontaneous symmetry breaking here.

Let us now generalize this example by allowing the KG field to become complex, $\varphi\colon M \to \mathbf{C}$, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\varphi\,\partial^\alpha\varphi^* - (-\frac{1}{2}a^2|\varphi|^2 + \frac{1}{4}b^2|\varphi|^4). \tag{15}$$

We assume $a \geq 0$, $b \geq 0$. This Lagrangian still admits the Poincaré symmetry, but the discrete Z_2 symmetry has been enhanced into a continuous $U(1)$ symmetry. Indeed, it is pretty obvious that the transformation

$$\varphi \to e^{i\alpha}\varphi, \quad \alpha \in \mathbf{R} \tag{16}$$

is a symmetry of $\mathcal{L}$. If you graph this potential in (x, y, z) space with $x = \Re(\varphi)$, $y = \Im(\varphi)$ and $z = V$, then you will see that the graph of the double well potential has been extended into a surface of revolution about z with the resulting shape being of the famous "Mexican hat" form. From this graphical point of view, the $U(1)$ phase symmetry of the Lagrangian specializes to symmetry of the graph of the potential with respect to rotations in the x-y plane.

Let us again consider the simplest possible states of the field, namely, the ones which admit the whole Poincaré group as a symmetry group. These field configurations are necessarily constants, and you can easily check that constant solutions must be critical points of the potential viewed as a function in the complex plane. So, $\varphi = const.$ is a solution to the field equations if and only if

$$\frac{b^2}{4}\varphi|\varphi|^2 - \frac{1}{2}a^2\varphi = 0.$$

There is an isolated solution $\varphi = 0$, and (now assuming $b > 0$) a continuous family of solutions characterized by

$$|\varphi| = \frac{a}{b}. \tag{17}$$

The solution $\varphi = 0$ "sits" at the local maximum at the top of the "hat". The solutions (17) sit at the circular set of global minima. As you might expect, the transformation (16) maps the solutions (17) among themselves. To see this explicitly, write the general form of φ satisfying (17) as

$$\varphi = \frac{a}{b}e^{i\theta}, \quad \theta \in \mathbf{R}. \tag{18}$$

The $U(1)$ symmetry transformation (16) then corresponds to $\theta \to \theta + \alpha$. The $U(1)$ transformation is a symmetry of the state $\varphi = 0$. Thus the solution $\varphi = 0$ has more symmetry than the family of solutions characterized by (17).

The stability analysis of these highly symmetric states of the complex scalar field generalizes from the double well example as follows. (I will spare you most of the details of the computations, but you might try to fill them in as a nice exercise.) In an inertial reference frame with coordinates $x^\alpha = (t, x^i)$, the conserved energy for this non-linear field is easily seen (from Noether's theorem) to be

$$E = \int d^3x \left(\frac{1}{2}|\varphi_{,t}|^2 + \frac{1}{2}\varphi_{,i}\,\varphi^{*i}_{,} - \frac{1}{2}a^2|\varphi|^2 + \frac{1}{4}b^2|\varphi|^4 \right). \tag{19}$$

You can easily check that the solutions given by $|\varphi| = 0, \frac{a}{b}$ are critical points of this energy functional and represent the lowest possible energy states. As before, the maximally symmetric state $\varphi = 0$ is unstable. The circle's worth of states (17) are quasi-stable in the following sense. Any displacement in field space yields a non-negative change in energy. To see this, write

$$\varphi = \rho e^{i\Theta}, \tag{20}$$

where ρ and Θ are spacetime functions. The energy takes the form

$$E = \int d^3x \left(\frac{1}{2}\rho_{,t}^2 + \frac{1}{2}\rho_{,i}\,\rho_{,}{}^{i} + \frac{1}{2}\rho^2(\Theta_{,t}^2 + \Theta_{,i}\,\Theta_{,}{}^{i}) - \frac{1}{2}a^2\rho^2 + \frac{1}{4}b^2\rho^4 \right). \tag{21}$$

The critical points of interest lie at $\rho = \frac{a}{b}$, $\Theta = const.$ Expanding the energy in displacements $(\delta\rho, \delta\Theta)$ from equilibrium yields

$$E = -\frac{1}{4}\frac{a^4}{b^2} + \int d^3x \left(\frac{1}{2}\delta\rho_{,t}^2 + \frac{1}{2}\delta\rho_{,i}\,\delta\rho_{,}{}^{i} + \frac{1}{2}\left(\frac{a}{b}\right)^2(\delta\Theta_{,t}^2 + \delta\Theta_{,i}\,\delta\Theta_{,}{}^{i}) + a^2\delta\rho^2 \right) \tag{22}$$

Evidently, all displacements *except* $\delta\rho = 0$, $\delta\Theta = const.$ increase the energy. The displacement $\delta\rho = 0$, $\delta\Theta = const.$ does not change the energy, as you might have guessed. The states (17) are the lowest energy states – the ground states. Thus the lowest energy is infinitely degenerate – the set of ground states (17) is topologically a circle. That these stable states have less symmetry than the unstable state will have some physical ramifications which we will understand after we take a little detour.

Dynamics near equilibrium

A significant victory for classical mechanics is the complete characterization of motion near stable equilibrium in terms of normal modes and characteristic frequencies of vibration. It is possible to establish analogous results in classical field theory via the process of *linearization*. This is even useful when one considers the associated quantum field theory: one can interpret the linearization of the classical field equations as characterizing particle states in the Fock space built on the vacuum state whose classical limit corresponds to the ground state about which one linearizes. If this seems a little too much to digest, that's ok – the point of this section is to make it easier to swallow.

Let us begin again with the simplest example: the real KG field with the double-well potential. Suppose that φ_0 is a given solution to the field equations. Any "nearby" solution we will denote by φ and we define the difference to be $\delta\varphi$.

$$\delta\varphi = \varphi - \varphi_0. \tag{23}$$

The field equation is the non-linear PDE:

$$\Box\varphi + a^2\varphi - b^2\varphi^3 = 0. \tag{24}$$

Using (23) we substitute $\varphi = \varphi_0 + \delta\varphi$. We then do 2 things: (1) we use the fact that φ_0 is a solution to the field equations; (2) we assume that $\delta\varphi$ is in some sense "small" so we can approximate the field equations in the vicinity of the given solution φ_0 by ignoring quadratic and cubic terms in $\delta\varphi$. We thus get the field equation linearized about the solution φ_0:

$$\Box\delta\varphi + (a^2 + 3b^2\varphi_0^2)\delta\varphi = 0. \tag{25}$$

This result can be obtained directly from the variational principle.

PROBLEM: Using (23) expand the action functional to quadratic order in $\delta\varphi$. Show that this approximate action, viewed as an action functional for the displacement field $\delta\varphi$, has (25) as its Euler-Lagrange field equation.

Evidently, the linearized equation (25) is a linear PDE for the displacement field $\delta\varphi$. If the given solution φ_0 is a constant solution the linearized PDE is mathematically identical to a Klein-Gordon equation for $\delta\varphi$ with mass given by $(a^2 + 3b^2\varphi_0^2)$. This is in fact the physical interpretation in the vicinity of one of the ground states we have been studying: the dynamics of the field is approximately that of a free KG field with mass $2a$.

From the way the linearized equation is derived, you can easily see that any displacement field $\delta\varphi$ constructed as an infinitesimal symmetry of the field equations will automatically satisfy the linearized equations. Indeed, this fact is the defining property of an infinitesimal symmetry. Here is a simple example.

PROBLEM: Consider time translations $\varphi(t, x, y, z) \to \tilde{\varphi} = \varphi(t+\lambda, x, y, z)$. Compute the infinitesimal form $\delta\varphi$ of this transformation as a function of φ and its derivatives. Show that if φ solves the field equation coming from (9) then $\delta\varphi$ solves the linearized equation (25).

These results easily generalize to the $U(1)$-invariant complex scalar field case, but a new and important feature emerges which leads to an instance of (the classical limit of) a famous result known as "Goldstone's theorem". Let us go through it carefully.

Things will be most transparent in the polar coordinates (20). The Lagrangian density takes the form

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\rho\,\partial^\alpha\rho - \frac{1}{2}\rho^2\partial_\alpha\Theta\,\partial^\alpha\Theta + \frac{1}{2}a^2\rho^2 - \frac{1}{4}b^2\rho^4. \tag{26}$$

The EL equations take the form

$$\Box\rho - \rho\partial_\alpha\Theta\,\partial^\alpha\Theta + a^2\rho - b^2\rho^3 = 0, \tag{27}$$

$$\partial_\alpha(\rho^2\partial^\alpha\Theta) = 0. \tag{28}$$

Two things to notice here. First, the symmetry under $\Theta \to \Theta + \alpha$ is apparent – only derivatives of Θ appear. Second, the associated conservation law is the content of (28).

Let us consider the linearization of these field equations about the circle's worth of equilibria $\rho = \frac{a}{b}$. We can proceed precisely as before, of course. But it will be instructive to perform the linearization at the level of the Lagrangian. To this end we write

$$\rho = \frac{a}{b} + \delta\rho;$$

we leave Θ as is since there is no equilibrium choice used for it. We expand the Lagrangian to quadratic order in $\delta\rho$ and Θ since that corresponds to linear field equations. We get

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\delta\rho\,\partial^\alpha\delta\rho - \frac{1}{2}\left(\frac{a}{b}\right)^2\partial_\alpha\Theta\,\partial^\alpha\Theta - \frac{1}{4}\frac{a^4}{b^2} - a^2\delta\rho^2 + \mathcal{O}(\delta\varphi^3).$$

Evidently, in the neighborhood of equilibrium the complex scalar field can be viewed as 2 real fields – no surprises so far. The main observation to be made is that one of the fields ($\delta\rho$) has a mass $m = a$ and one of the fields (Θ) is massless.

To get a feel for what just happened, let us consider a very similar $U(1)$ symmetric theory, just differing in the sign of the quadratic potential term in the Lagrangian. The Lagrangian density is

$$\mathcal{L}' = -\frac{1}{2}\partial_\alpha\varphi\,\partial^\alpha\varphi^* - \mu^2|\varphi|^2 - \frac{1}{4}b^2|\varphi|^4. \tag{29}$$

There is only a single Poincaré invariant critical point, $\varphi = 0$, which is a global minimum of the energy and which is also $U(1)$ invariant, so the $U(1)$ symmetry is *not* spontaneously broken in the ground state. In the vicinity of the ground state the linearized Lagrangian takes the simple form*

$$\mathcal{L}' = -\frac{1}{2}\partial_\alpha \delta\varphi\, \partial^\alpha \delta\varphi^* - \mu^2 |\delta\varphi|^2 + \mathcal{O}(\delta\varphi^3).$$

Here of course we have the Lagrangian of a complex-valued KG field $\delta\varphi$ with mass μ; equivalently, we have two real scalar fields with mass μ.

To summarize thus far: With a complex KG field described by a potential such that the ground state shares all the symmetries of the Lagrangian, the physics of the theory near the ground state is that of a pair of real, massive KG fields. Using instead the Mexican hat potential, the ground state of the complex scalar field does not share all the symmetries of the Lagrangian – there is spontaneous symmetry breaking – and the physics of the field theory near equilibrium is that of a pair of scalar fields, one with mass and one which is massless.

To some extent, it is not too hard to understand *a priori* how these results occur. In particular, we can see why a massless field emerged from the spontaneous symmetry breaking. For Poincaré invariant solutions – which are constant functions in spacetime – the linearization of the field equations involves:

(1) the derivative terms, which being quadratic in the fields, and given the ground state is constant, are the same in the linearization as for the full field equations;

(2) the Taylor expansion of the potential $V(\varphi)$ to second order about the constant equilibrium solution φ_0.

Because of (1), the nature of the mass terms comes from the expansion (2). Because of the symmetry of the Lagrangian, and because of the spontaneous symmetry breaking, we are guaranteed that through each point in the set of field values there will be a curve (with tangent vector given by the infinitesimal symmetry) along which the potential will not change. Taylor expansion about the ground state in this symmetry direction will yield only vanishing contributions because the potential has vanishing derivatives in that direction. Thus the broken symmetry direction(s) defines the direction(s) in field space which correspond to massless fields. This is the essence of the (classical limit of the) Goldstone theorem: to each broken continuous symmetry generator there is a massless field.

The Abelian Higgs model

The Goldstone result in conjunction with minimal coupling to an electromagnetic field yields a very important new behavior known as the "Higgs phenomenon". This results

* We do not use polar coordinates which are ill-defined at the origin.

from the interplay of the spontaneously $U(1)$ symmetry and the local gauge symmetry. We start with a charged self-interacting scalar field coupled to the electromagnetic field; the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \mathcal{D}^\alpha\varphi^*\mathcal{D}_\alpha\varphi - V(\varphi). \tag{30}$$

We will again choose the potential so that spontaneous symmetry breaking occurs:

$$V(\varphi) = -\frac{1}{2}a^2|\varphi|^2 + \frac{1}{4}b^2|\varphi|^4. \tag{31}$$

To see what happens in detail, we return to the polar coordinates (20). The Lagrangian takes the form:

$$\mathcal{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{2}\partial_\alpha\rho\partial^\alpha\rho - \frac{1}{2}\rho^2(\partial_\alpha\Theta + eA_\alpha)(\partial^\alpha\Theta + eA^\alpha) + \frac{1}{2}a^2\rho^2 - \frac{1}{4}b^2\rho^4. \tag{32}$$

The Poincaré invariant ground state(s) can be determined as follows. As we have observed, a Poncaré invariant function φ is necessarily a constant. Likewise, it is too not hard to see that the only Poincaré invariant (co)vector is the the zero (co)vector $A_\alpha = 0$. Consequently, the Poincaré invariant ground state, as before, is specified by

$$\rho = \frac{a}{b}, \quad \Theta = const. \quad A_\alpha = 0. \tag{33}$$

As before the $U(1)$ symmetry of the theory is not a symmetry of this state. As before, we want to expanding to quadratic order about the ground state. To this end we write

$$A_\alpha = 0 + \delta A_\alpha, \quad \rho = \frac{a}{b} + \delta\rho, \quad \Theta = \Theta_0 + \delta\Theta,$$

where Θ_0 is a constant. We also define

$$B_\alpha = \delta A_\alpha + \frac{1}{e}\partial_\alpha\delta\Theta.$$

Ignoring terms of cubic and higher order in the displacements $(\delta A, \delta\rho, \delta\Theta)$ we then get

$$\begin{aligned}\mathcal{L} \approx &-\frac{1}{4}(\partial_\alpha B_\beta - \partial_\beta B_\alpha)(\partial^\alpha B^\beta - \partial^\beta B^\alpha) - \frac{1}{2}\left(\frac{ae}{b}\right)^2 B_\alpha B^\alpha \\ &-\frac{1}{2}\partial_\alpha\delta\rho\,\partial^\alpha\delta\rho - \frac{1}{2}a^2\delta\rho^2\end{aligned} \tag{34}$$

PROBLEM: Starting from (30) derive the results (32) and (34).

As you can see, excitations of ρ around the ground state are those of a scalar field with mass a, as before. To understand the rest of the Lagrangian we need to understand the *Proca Lagrangian*:

$$\mathcal{L}_p = -\frac{1}{4}(\partial_\alpha B_\beta - \partial_\beta B_\alpha)(\partial^\alpha B^\beta - \partial^\beta B^\alpha) - \frac{1}{2}\kappa^2 B_\alpha B^\alpha. \tag{35}$$

For $\kappa = 0$ this is just the usual electromagnetic Lagrangian. Otherwise...

PROBLEM: Assuming $\kappa \neq 0$, show that the Euler-Lagrange equations for B_α defined by (35) are equivalent to

$$(\Box - \kappa^2)B_\alpha = 0, \quad \partial^\alpha B_\alpha = 0.$$

Thus each component of B_α behaves as a Klein-Gordon field with mass κ. (The divergence condition means that there are only 3 independent scalar fields in play.) Thus the "Proca field theory" is a model for an electromagnetic field with mass. The quantum theory of the Proca field is the simplest way to model "massive photons". Notice that the Proca theory does not admit the gauge symmetry of electromagnetism.

The punchline here is that spontaneous symmetry breaking coupled with gauge symmetry leads to a field theory whose dynamics near the ground state is that of a massive scalar and a massive vector field. This is the simplest instance of the "Higgs phenomenon".

It is interesting to ask what became of the massless "Goldstone boson" which appeared when we studied spontaneous symmetry breaking of the scalar field without the gauge field. The answer is that the direction in field space (the Θ direction) which would have corresponded to that massless field is now a direction corresponding the the $U(1)$ gauge symmetry. Indeed, for any particular Θ field one could always adjust the gauge of A_α via

$$A_\alpha \to A_\alpha + \partial_\alpha \Lambda$$

and remove Θ from the Lagrangian and field equations. So, there is a sense in which the Θ degrees of freedom are "pure gauge" and do not appear at all!

Finally, we point out that the Higgs phenomenon can be generalized considerably. Without going into detail, we sketch the generalization. Consider a system of fields with symmetry group G and ground state which breaks that symmetry. Couple these fields to gauge fields with gauge group which includes G. Excitations of the theory near the ground state will have the gauge fields corresponding to G acquiring a mass. This is precisely how the W and Z bosons of the weak interaction acquire their effective masses at (relatively) low energies.

PROBLEMS

1. It is not true that every symmetry of the field equations is a symmetry of the Lagrangian. Consider the massless KG field. Show that the scaling transformation $\tilde{\varphi} = (const.)\varphi$ is a symmetry of the field equations but is *not* a symmetry of the Lagrangian.

2. Using (23) expand the action functional to quadratic order in $\delta\varphi$. Show that this approximate action, viewed as an action functional for the displacement field $\delta\varphi$, has (25) as its Euler-Lagrange field equation.

3. Consider time translations $\varphi(t, x, y, z) \to \tilde{\varphi} = \varphi(t+\lambda, x, y, z)$. Compute the infinitesimal form $\delta\varphi$ of this transformation as a function of φ and its derivatives. Show that if φ solves the field equation coming from (9) then $\delta\varphi$ solves the linearized equation (25).

4. Starting from (30) derive the results (32) and (34).

5. Assuming $\kappa \neq 0$, show that the Euler-Lagrange equations for B_α defined by (35) are equivalent to

$$(\square - \kappa^2)B_\alpha = 0, \quad \partial^\alpha B_\alpha = 0.$$

6. Derive the result (5).