Scalar Electrodynamics. The principle of local gauge invariance. Lower-degree conservation laws.

Scalar Electrodynamics

Let us now explore an introduction to the field theory called *scalar electrodynamics*, in which one considers a coupled system of Maxwell and charged KG fields. There is an infinite number of ways one could try to couple these fields. There is essentially only one physically interesting way, and this is the one we shall be exploring. Mathematically, too, this particular coupling has many interesting features which we shall explore. Strictly speaking, one cannot derive these equations – we can only postulate them. However, it is possible to provide some fundamental physical motivation for the postulated form of scalar electrodynamics, which we shall now try to do. It is easiest to proceed via Lagrangians. For simplicity we will restrict attention to flat spacetime in inertial Cartesian coordinates, but our treatment is easily generalized to an arbitrary spacetime in a coordinate-free way.

Let us return to the electromagnetic theory, but now with electrically charged sources. Recall that if $j^\alpha(x)$ is some *given* divergence-free vector field on spacetime, representing some externally specified charge-current distribution, then the response of the electromagnetic field to the given source is dictated by the Lagrangian

$$\mathcal{L}_j = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + j^\alpha A_\alpha. \tag{1}$$

Incidentally, given the explicit appearance of A_α, one might worry about the gauge symmetry of this Lagrangian. But it is easily seen that the gauge transformation is a divergence symmetry of this Lagrangian. Indeed, under a gauge transformation of the offending term we have

$$\begin{aligned} j^\alpha A_\alpha \longrightarrow & j^\alpha(A_\alpha + \partial_\alpha \Lambda) \\ &= j^\alpha A_\alpha + D_\alpha(\Lambda j^\alpha), \end{aligned} \tag{2}$$

where we had to use the fact

$$\partial_\alpha j^\alpha(x) = 0. \tag{3}$$

The idea now is that we don't want to specify the sources in advance, we want the theory to tell us how they behave. In other words, we want to include the sources as part of the dynamical variables of our theory. In all known instances the correct way to do this always follows the same pattern: the gauge fields affect the "motion" of the sources, and the sources affect the form of the gauge field. Here we will use the electromagnetic field as the gauge field and charged ($U(1)$ symmetric) KG field as the source. The reasoning for this latter choice goes as follows.

Since this is a course in field theory, we are required to only use fields to model things like electrically charged matter, so we insist upon a model for the charged sources built from a classical field. So, we need a classical field theory that admits a conserved current that we can interpret as an electric 4-current. The KG field admitted 10 conserved currents corresponding to conserved energy, momentum and angular momentum. But we know that the electromagnetic field is not driven by such quantities, so we need another kind of current. To find such a current we turn to the charged KG field. In the absence of any other interactions, this field admits the conserved current

$$j^\alpha = -ig^{\alpha\beta}\left(\varphi^*\varphi_{,\beta} - \varphi\varphi^*_{,\beta}\right). \tag{4}$$

The simplest thing to try is to build a theory in which this is the current that drives the electromagnetic field. This is the correct idea, but the most naive attempt to implement this strategy falls short of perfection. To see this, imagine a Lagrangian of the form

$$\mathcal{L}_{wrong} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - (\partial^\alpha\varphi^*\partial_\alpha\varphi + m^2|\varphi|^2) - iA_\alpha g^{\alpha\beta}\left(\varphi^*\varphi_{,\beta} - \varphi\varphi^*_{,\beta}\right). \tag{5}$$

The idea is that the EL equations for A will give the Maxwell equations with the KG current as the source. The KG EL equations will now involve A, but that is ok since we expect the presence of the electromagnetic field to affect the sources. But here is one big problem with this Lagrangian: it is no longer gauge invariant! Recall that the gauge invariance of the Maxwell Lagrangian with prescribed sources made use of the fact that the current was divergence-free. But now the current is divergence-free, not identically, but only when the field equations hold. The key to escaping this difficulty is to let the KG field participate in the gauge symmetry. This forces us to modify the Lagrangian as we shall now discuss.

Minimal coupling: the gauge covariant derivative

The physically correct way to get a gauge invariant Lagrangian for the coupled Maxwell-KG theory, that still gives the $j^\alpha A_\alpha$ kind of coupling is rather subtle and clever. Let me begin by just stating the answer. Then I will try to show how it works and how one might even be able to derive it from some new, profound ideas. The answer is to modify the KG Lagrangian via "minimal coupling", in which one replaces

$$\partial_\alpha\varphi \to \mathcal{D}_\alpha\varphi := (\partial_\alpha + ieA_\alpha)\varphi, \tag{6}$$

and

$$\partial_\alpha\varphi^* \to \mathcal{D}_\alpha\varphi^* := (\partial_\alpha - ieA_\alpha)\varphi^*, \tag{7}$$

Here e is a parameter reflecting the coupling strength between the charged field φ and the gauge field. It is a *coupling constant*. In a more correct quantum field theory description e is the charge of a particle excitation of the quantum field φ. The effect of an electromagnetic field described by A_α upon the KG field is then described by the Lagrangian

$$\mathcal{L}_{KG} = -\mathcal{D}^\alpha\varphi^*\mathcal{D}_\alpha\varphi - m^2|\varphi|^2. \tag{8}$$

This Lagrangian yields field equations which involve the wave operator modified by terms built from the electromagnetic potential. These additional terms represent the effect of the electromagnetic field on the charged scalar field.

PROBLEM: Compute the EL equations of $\mathcal{L}_{KG}$ in (8).

The Lagrangian (8) still admits the $U(1)$ phase symmetry, but now the conserved current is defined using both φ and A:

$$j^\alpha = -ie\left(\varphi^*\mathcal{D}^\alpha\varphi - \varphi\mathcal{D}^\alpha\varphi^*\right). \tag{9}$$

(We have normalized j^α with e to get the physically correct size and units for the conserved charge.) This Lagrangian and current depend explicitly upon A and so will not be gauge invariant unless we include a transformation of φ. We therefore modify the gauge transformation to be of the form

$$A_\alpha \longrightarrow A_\alpha + \partial_\alpha\Lambda, \tag{10}$$

$$\varphi \longrightarrow e^{-ie\Lambda}\varphi, \quad \varphi^* \longrightarrow e^{ie\Lambda}\varphi^*. \tag{11}$$

You can easily verify that under a gauge transformation we have the fundamental relation (which justifies the minimal coupling prescription)

$$\mathcal{D}_\alpha\varphi \longrightarrow e^{-ie\Lambda}\mathcal{D}_\alpha\varphi, \tag{12}$$

$$\mathcal{D}_\alpha\varphi^* \longrightarrow e^{ie\Lambda}\mathcal{D}_\alpha\varphi^*. \tag{13}$$

For this reason $\mathcal{D}_\alpha$ is sometimes called the *gauge covariant derivative.* There is a nice geometric interpretation of this covariant derivative, which we shall discuss later. For now, because of this "covariance" property of $\mathcal{D}$ we have that the Lagrangian and current are gauge invariant. The Lagrangian for scalar electrodynamics is now

$$\mathcal{L}_{SED} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \mathcal{D}^\alpha\varphi^*\mathcal{D}_\alpha\varphi - m^2|\varphi|^2. \tag{14}$$

We now note some important structural features of this Lagrangian. If we expand out all the gauge covariant derivatives we see that

$$\mathcal{L}_{SED} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \partial^\alpha\varphi^*\partial_\alpha\varphi - m^2|\varphi|^2 - (ie)A_\alpha\left(\varphi^*\partial^\alpha\varphi - \varphi\partial^\alpha\varphi^* + 2ieA^\alpha|\varphi|^2\right). \tag{15}$$

This Lagrangian is the sum of the electromagnetic Lagrangian, the free charged KG Lagrangian, and a $j \cdot A$ "interaction term". The vector field contracted with A_α is almost the conserved current j^α, except for the last term involving the square of the gauge field which is needed for invariance under the gauge transformation (11) and for the current to be conserved (modulo the field equations). The EL equations for the Maxwell field are of the desired form:

$$\partial_\beta F^{\alpha\beta} = -j^\alpha, \tag{16}$$

where the current is defined using the covariant derivative instead of the ordinary derivative:

$$j^\alpha = -ie\left(\varphi^* \mathcal{D}^\alpha \varphi - \varphi \mathcal{D}^\alpha \varphi^*\right). \tag{17}$$

Thus we have solved the gauge invariance problem *and* obtained a consistent version of the Maxwell equations with conserved sources using the minimal coupling prescription.

One more feature to ponder: the charged current serving as the source for the Maxwell equations is built from the KG field *and the Maxwell field.* Physically this means that one cannot say the charge "exists" only in the KG field. In an interacting system the division between source fields and fields mediating interactions is somewhat artificial. This is physically reasonable, if perhaps a little unsettling. Mathematically, this feature stems from the demand of gauge invariance. Just like the vector potential, the KG field is no longer uniquely defined - it is subject to a gauge transformation as well! In the presence of interaction, the computation of the electric charge involves a gauge invariant combination of the KG and electromagnetic field. To compute, say, the electric charge contained in a volume V one should take a solution (A, φ) of the coupled Maxwell-KG equations and substitute it into

$$Q_V = \int_V d^3x\, ie\left(\varphi^* \mathcal{D}^0 \varphi - \varphi \mathcal{D}^0 \varphi^*\right). \tag{18}$$

This charge is conserved and gauge invariant.

Global and Local Symmetries

We have constructed the Lagrangian for scalar electrodynamics. The key step was to introduce the coupling between the Maxwell field A and the charged KG field by replacing in the KG Lagrangian the ordinary derivative with the gauge covariant derivative. With this replacement, the coupled KG-Maxwell theory is defined by adding the modified KG Lagrangian to the electromagnetic Lagrangian. There is a rather deep way of viewing this construction which we shall now explore.

Let us return to the free, charged KG theory, described by the Lagrangian

$$\mathcal{L}^0_{KG} = -(\varphi^*{,}^\alpha \varphi_{,\alpha} + m^2|\varphi|^2). \tag{19}$$

This field theory admits a conserved current

$$j^\alpha = -iq(\varphi^* \varphi,^\alpha - \varphi\varphi^*,^\alpha), \tag{20}$$

which we want to interpret as corresponding to a conserved electric charge "stored" in the field. Of course, the presence of electric charge in the universe only manifests itself by virtue of its electromagnetic interactions. How should the conserved charge in the KG field be interacting? Well, we followed one rather *ad hoc* path to introducing this interaction in the last lecture. Let us revisit the construction with a focus upon symmetry considerations, which will lead to a very profound way of interpreting and systematizing the construction.

The current j^α is conserved because of the *global* $U(1)$ phase symmetry. For any $\alpha \in \mathbf{R}$ this symmetry transformation is

$$\varphi \to e^{-iq\alpha}\varphi, \quad \varphi^* \to e^{iq\alpha}\varphi^*, \tag{21}$$

where q is a parameter to be fixed by experimental considerations, as we shall see. This transformation shifts the phase of the scalar field by the amount α everywhere in space and for all time. This is why the transformation is called "global". You can think of this global phase assignment as a sort of internal reference frame for the charged field. Nothing depends upon this phase – the choice of reference frame is just a matter of convenience and/or convention. That is why a global change of phase is a symmetry of the theory. This is completely analogous to the use/choice of an inertial reference frame in spacetime physics according to the principles of the special theory of relativity – a global change of spacetime inertial reference frame is a symmetry of special relativistic theories.

The presence of the electromagnetic interaction can be seen as a "localizing" or "gauging" of this global symmetry so that one can is free to redefine the phase of the field independently at each spacetime event (albeit smoothly). This "general relativity" of phase is accomplished by demanding that the theory be modified so that one has the symmetry

$$\varphi \to e^{-iq\alpha(x)}\varphi, \quad \varphi^* \to e^{iq\alpha(x)}\varphi^*, \tag{22}$$

where $\alpha(x)$ is any function on the spacetime manifold M. Of course, the original Lagrangian $\mathcal{L}^0_{KG}$ fails to have this *local* $U(1)$ transformation as a symmetry since,

$$\partial_\mu(e^{-iq\alpha(x)}\varphi) = e^{-iq\alpha(x)}\varphi,_\mu - iqe^{-iq\alpha(x)}\varphi\alpha,_\mu . \tag{23}$$

However, we can introduce a *gauge field* A_μ, which transforms by

$$A_\mu \longrightarrow A_\mu + \partial_\mu\alpha, \tag{24}$$

and then introduce the *covariant derivative*

$$\mathcal{D}_\alpha\varphi := (\partial_\alpha + iqA_\alpha)\varphi, \tag{25}$$

and

$$\mathcal{D}_\alpha \varphi^* := (\partial_\alpha - iqA_\alpha)\varphi^*, \tag{26}$$

which satisfies

$$\mathcal{D}_\mu(e^{-iq\alpha(x)}\varphi) = e^{-iq\alpha(x)}\mathcal{D}\varphi. \tag{27}$$

Then with the Lagrangian modified via

$$\partial_\mu \varphi \to \mathcal{D}_\mu \varphi, \tag{28}$$

so that

$$\mathcal{L}_{KG} = -\mathcal{D}^\alpha \varphi^* \mathcal{D}_\alpha \varphi - m^2|\varphi|^2 \tag{29}$$

we get the local $U(1)$ symmetry, as shown previosly. Thus the minimal coupling rule that we invented earlier can be seen as a way of turning the global $U(1)$ symmetry into a local $U(1)$ gauge symmetry. One can also get the satisfying mental picture that the electromagnetic interaction of charges is the principal manifestation of this local phase symmetry in nature.

The electromagnetic interaction of charges is described mathematically by the $\partial_\alpha \to \mathcal{D}_\alpha$ prescription described above. But the story is not complete since we have not given a complete description of the electromagnetic field itself. We need to include the electromagnetic Lagrangian into the total Lagrangian for the system. How should we think about the electromagnetic Lagrangian from the point of view the principle of local gauge invariance? The electromagnetic Lagrangian is the simplest scalar that can be made from the field strength tensor. The field strength tensor itself can be viewed as the "curvature" of the gauge covariant derivative, computed via the commutator:

$$\left(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu\right)\varphi = iqF_{\mu\nu}\varphi. \tag{30}$$

From this relation it follows immediately that F is gauge invariant.

PROBLEM: Verify the result (30).

Thus the electromagnetic Lagrangian

$$\mathcal{L}_{EM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{31}$$

admits the local $U(1)$ symmetry and can be added to the locally invariant KG Lagrangian to get the total Lagrangian for the theory

$$\mathcal{L}_{SED} = \mathcal{L}_{KG} + \mathcal{L}_{EM}. \tag{32}$$

In this way we have an interacting theory designed by local $U(1)$ gauge symmetry. The parameter q, which appears via the gauge covariant derivative, is a "coupling constant" and characterizes the strength with which the electromagnetic field couples to the charged aspect of the KG field. In the limit in which $q \to 0$ the theory becomes a decoupled juxtaposition of the non-interacting (or "free") charged KG field theory and the non-interacting (free) Maxwell field theory. In principle, the parameter q is determined by suitable experiments.

By construction, the theory of scalar electrodynamics still admits the global $U(1)$ symmetry, with $\alpha = const.$

$$\varphi \to e^{-iq\alpha}\varphi, \quad \varphi^* \to e^{iq\alpha}\varphi^*, \tag{33}$$

$$A_\mu \to A_\mu, \tag{34}$$

as a Lagrangian symmetry. Infinitesimally we can write this transformation as

$$\delta\varphi = -iq\alpha\varphi, \quad \delta\varphi^* = iq\alpha\varphi^*, \tag{35}$$

$$\delta A_\mu = 0. \tag{36}$$

As we have seen, this leads to the conserved Noether current

$$j^\alpha = -iq\left(\varphi^*\mathcal{D}^\alpha\varphi - \varphi\mathcal{D}^\alpha\varphi^*\right), \tag{37}$$

corresponding to the conserved electric charge.

PROBLEM: Derive the formula (37) for j^α.

This is the current that serves as source for the Maxwell field. The presence of the gauge field in the current is needed so that the Noether current is suitably "gauge invariant", that is, insensitive to the local $U(1)$ transformation. It also reflects the fact that the equations of motion for φ, which must be satisfied in order for the current to be conserved, depend upon the Maxwell field as is appropriate since the electromagnetic field affects the motion of its charged sources.

By construction, the theory of scalar electrodynamics admits the local $U(1)$ gauge symmetry. With $\alpha(x)$ being any function, the symmetry is

$$\varphi \to e^{-iq\alpha(x)}\varphi, \quad \varphi^* \to e^{iq\alpha(x)}\varphi^*,$$

$$A_\mu \to A_\mu + \partial_\mu\alpha(x). \tag{38}$$

There is a corresponding Noether identity. To compute it we consider an infinitesimal gauge transformation:

$$\delta\varphi = -iq\alpha(x)\varphi, \quad \delta\varphi^* = iq\alpha(x)\varphi^*, \tag{39}$$

$$\delta A_\mu = \partial_\mu \alpha(x). \tag{40}$$

Since the Lagrangian is gauge invariant we have the identity

$$0 = \delta\mathcal{L} = \mathcal{E}_\varphi(-iq\alpha(x)\varphi) + \mathcal{E}_{\varphi^*}(iq\alpha(x)\varphi^*) + \mathcal{E}^\mu(\partial_\mu\alpha(x)) + \text{divergence}, \tag{41}$$

where $\mathcal{E}$ denotes the various Euler-Lagrange expressions. Since this must hold for arbitrary $\alpha(x)$ we get the Noether identity

$$D_\mu\mathcal{E}^\mu + iq(\varphi\mathcal{E}_\varphi - \varphi^*\mathcal{E}_{\varphi^*}) = 0. \tag{42}$$

On the other hand, the terms involving the EL expressions for the KG field are the same as would arise in the identity associated with the global gauge symmetry:

$$D_\mu j^\mu + iq(\varphi\mathcal{E}_\varphi - \varphi^*\mathcal{E}_{\varphi^*}) = 0. \tag{43}$$

Thus we have the equivalent identity

$$D_\mu(\mathcal{E}^\mu - j^\mu) = 0. \tag{44}$$

A lower-degree conservation law

The conserved current for scalar electrodynamics,

$$j^\alpha = -iq\left(\varphi^*\mathcal{D}^\alpha\varphi - \varphi\mathcal{D}^\alpha\varphi^*\right), \tag{45}$$

features in the Maxwell equations $\Delta = 0$ via:

$$\Delta^\alpha = F^{\alpha\beta}_{,\alpha} - j^\alpha. \tag{46}$$

It follows that j^α is a "trivial" conservation law!

$$j^\alpha = F^{\alpha\beta}_{,\alpha} - \Delta^\alpha. \tag{47}$$

This result is intimately related to the fact that the charge contained in a given spatial region can be computed using just electromagnetic data on the surface bounding that region. Indeed, the conserved electric charge in a 3-dimensional spacelike region V at some time $x^0 = const.$ is given by

$$Q_V = \int_V dV\, j^0 = \int_V dV\, F^{i0}_{,i} = \int_S dS\, \hat{n}\cdot\vec{E}, \tag{48}$$

where $S = \partial V$ and $E^i = F^{0i}$ is the electric field in the inertial frame with time $t = x^0$.

A more geometric way to characterize this relationship is via differential forms and Stokes theorem. Recall that if ω is a differential p-form and V is a $p+1$-dimensional region with p-dimensional boundary S (*e.g.,* V is the interior of a 2-sphere and S is the 2-sphere, then Stokes theorem says

$$\int_V d\omega = \int_S \omega. \tag{49}$$

This generalizes the version of Stokes theorem you learned in multi-variable calculus in Euclidean space to manifolds in any dimension. If we let $\omega = \star F$ then from Stokes theorem we have

$$Q_V = \int_S \star F.$$

An important application of Stokes theorem we will need goes as follows. Let Q be the p-form ω integrated over the p-dimensional space $S = \partial V$,

$$\chi = \int_S \omega. \tag{50}$$

Consider any deformation of S into a new surface $S' = \partial V'$ and let Q' be the integral of ω over that space:

$$Q' = \int_{S'} \omega. \tag{51}$$

The relation between these 2 quantities can be obtained using Stokes theorem:

$$Q' - Q = \int_{S'} \omega - \int_S \omega = \int_{S'-S} \omega = \int_V d\omega,$$

where $\partial V = S' - S$. In particular, if ω is a *closed p-form*, that is, $d\omega = 0$, then $Q = Q'$ and the integral Q is independent of the choice of the space S.

As we have mentioned before, we can view the electromagnetic tensor as a 2-form F via

$$F = F_{\alpha\beta} dx^\alpha \otimes dx^\beta = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta. \tag{52}$$

Using the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$, we can construct the *Hodge dual* $\star F$, defined by

$$\star F = \frac{1}{2} (\star F)_{\alpha\beta} dx^\alpha \wedge dx^\beta, \tag{53}$$

where

$$(\star F)_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}. \tag{54}$$

In terms of F and $\star F$ the Maxwell equations read

$$dF = 0, \quad d \star F = J, \tag{55}$$

where J is the Hodge dual of the electric current:

$$J = \frac{1}{3!}\epsilon_{\alpha\beta\gamma\delta}j^\delta dx^\alpha \wedge dx^\beta \wedge dx^\gamma. \tag{56}$$

The conservation of electric current is normally expressed in terms of a closed 3-form: $dJ = 0$ modulo the field equations. As we mentioned, the field equations in fact tell us that the conservation law is "trivial" in the sense that $J = d\star F$. An alternative is to view the conservation law in terms of a closed 2-form $\star F$ as follows.

Evidently the integral of both F and $\star F$ over a closed surface is independent of any continuous deformation of that surface. Now, since $F = dA$, the integral of F over a closed surface vanishes by Stokes theorem, which you can check as a nice exercise. But the integral of $\star F$ over a closed spacelike surface at a given instant of time is the electric charge contained in that surface at that time. Suppose that charge is confined to a finite volume and that the surface encloses that volume. So long as we deform the volume in a region where J vanishes, then the integral is independent of the surface. This deformation could be in spatial directions at a given instant of time, or it could be a deformation corresponding to time evolution. The constancy of the surface integral under continuous time evolution is a way of viewing the conservation of electric charge in terms of a closed 2-form.

The existence of conservation laws of the traditional sort – divergence-free currents or closed 3-forms – is tied to the existence of symmetries via Noether's theorem. It is natural to ask if there is any symmetry-based origin to conserved 2-forms such as we have in electrodynamics with $\star F$. The answer is yes. Details would take us to far afield, but let me just mention that closed 2-forms (in 4 dimensions) arise in a field theory when (1) the theory admits a gauge symmetry, (2) every solution of the field equations admits a gauge transformation which fixes that solution. Now consider pure electromagnetism in a region of spacetime with no sources. Of course, criterion (1) is satisfied. To see that criterion (2) is satisfied consider the gauge transformation by a constant function.

Scalar electrodynamics and fiber bundles

There is a beautiful geometric interpretation of SED in terms of a famous mathematical structure called a fiber bundle. I debated with myself for a long time whether or not to try and describe this to you. I decided that I could not resist mentioning it, so that those of you who are so-inclined can get exposed to it. On the other hand, a complete presentation would take us too far afield and not everybody in this class is going to be properly prepared for a full-blown treatment. So if you don't mind I will just give a quick and dirty summary of the salient points. A more complete – indeed, a more correct – treatment can be found in many advanced texts. Of course, the problem is that to use these advanced texts takes a considerable investment in acquiring prerequisites. The idea of our brief discussion is to provide a first introductory step in that direction. A technical point: for simplicity in what follows we shall not emphasize the role of the gauge field as a connection on a *principal bundle*, but rather its role as defining a connection on associated vector bundles.

Recall that our charged KG field can be viewed as a mapping

$$\varphi\colon M \to \mathbf{C}. \tag{57}$$

We can view φ as a section of a fiber bundle

$$\pi\colon E \to M \tag{58}$$

where

$$\pi^{-1}(x) = \mathbf{C}, \quad x \in M. \tag{59}$$

The space $\pi^{-1}(x) \approx \mathbf{C}$ is the *fiber* over x. Since $\mathbf{C}$ is a vector space, this type of fiber bundle is called a *vector bundle*. For us, $M = \mathbf{R}^4$ and it can be shown that there is always a diffeomorphism that makes the identification:

$$E \approx M \times \mathbf{C}. \tag{60}$$

From this you can see how the fancy bundle description just encodes our usual set up. Recall that a *cross section* of E is a map

$$\varphi\colon M \to E \tag{61}$$

satisfying

$$\pi \circ \varphi = id_M, \tag{62}$$

which we can identify with our KG field via

$$x \to (x, \varphi(x)). \tag{63}$$

Thus, given the identification $E \approx M \times \mathbf{C}$ we see that the bundle point of view just describes the geometric setting of our theory: complex valued functions on $\mathbf{R}^4$. To some extent, the most interesting issue is that this identification is far from unique. Let us use coordinates (x^α, z) for E. Each set of such coordinates provides an identification of E with $M \times \mathbf{C}$. Since we use a fixed (flat) metric on M, one can restrict attention to inertial Cartesian coordinates on M, in which case one can only redefine x^α by a Poincaré transformation. What is more interesting for us in this discussion is the freedom to redefine the way that the complex numbers are "glued" to each spacetime event. Recall that to build the charged KG field we also had to pick a scalar product on the vector space $\mathbf{C}$; of course we just used the standard one

$$(z, w) = z^* w. \tag{64}$$

We can therefore restrict attention to linear changes of our coordinates on **C** which preserve this scalar product. This leads to the allowed changes of fiber coordinates being just the phase transformations

$$z \to e^{i\alpha} z. \tag{65}$$

We can make this change of coordinates on **C** for each fiber so that on $\pi^{-1}(x)$ we make the transformation

$$z \to e^{i\alpha(x)} z. \tag{66}$$

There is no intrinsic way to compare points on different fibers, and this fact reflects itself in the freedom to redefine our labeling of those points in a way that can vary from fiber to fiber. We have seen this already; the change of fiber coordinates $z \to e^{i\alpha(x)}z$. corresponds to the gauge transformation of the charged KG field:

$$\varphi \to e^{\alpha(x)}\varphi. \tag{67}$$

When building a field theory of the charged KG field we need to take derivatives. Now, to take a derivative means to compare the value of φ at two neighboring points on M. From our fiber bundle point of view, this means comparing points on two different fibers, which we have just pointed out is rather arbitrary. Put differently, there is no "natural" way to differentiate a section of a fiber bundle. This is why, as we saw, the ordinary derivative of the KG field does not transform homogeneously under a gauge transformation. Thus, for example, to say that a KG field is a constant, $\partial_\alpha\varphi = 0$ is not an intrinsic statement since a change in the bundle coordinates will negate it.

A definition of derivative involves the introduction of additional structure beyond the manifold and metric. (One often introduces this structure implicitly!) This additional structure is called a *connection* and the resulting notion of derivative is called the *covariant derivative* defined by the section. A connection can be viewed as a definition of how to compare points on neighboring fibers. If you are differentiating in a given direction, the derivative will need to associate to that direction a linear linear transformation (actually, a phase transformation) which "aligns" the vector spaces/fibers and allows us to compare them. Since the derivative involves an infinitesimal motion in M, it turns out that this fiber transformation is infinitesimal, and since the fibers are fixed up to the phase transformation, an infinitesimal version involves multiplication by a pure imaginary number (think: $e^{i\alpha} = 1 + i\alpha + \ldots$). So, at each point $x \in M$ a a connection can be specified by a pure imaginary-valued 1-form, which we write as $iqA_\alpha(x)$. The covariant derivative is then

$$\mathcal{D}_\alpha\varphi = (\partial_\alpha + iqA_\alpha)\varphi. \tag{68}$$

You can see that the role of the connection A is to adjust the correspondence between fibers relative to that provided by the given choice of coordinates so that the rate of change of a function in a given direction is governed by the coordinate rate of change plus the correction factor A that provides the chosen definition of how to decide when one is moving up and down the fibers as one moves along M.

As we have seen, if we make a redefinition of the coordinates on each fiber by a local gauge transformation, then we must correspondingly redefine the 1-form via

$$z \to e^{i\alpha(x)} z, \quad A_\alpha \to A_\mu + \partial_\mu \alpha. \tag{69}$$

This guarantees that the covariant derivative transforms homogeneously under a gauge transformation in the same way that the KG field itself does. Thus, in particular, if a KG field is constant with the given choice of connection then this remains true in any (fiber) coordinates.

The connection, and the covariant derivative, are defined by a choice of the 1-form A_α. If we fix this 1-form once and for all, then we are not free to make gauge transformations – since these coordinate transformations will not in general fix the connection – and we have the theory of a charged KG field in a prescribed Maxwell field. In particular, if we choose $A_\mu = 0$ we recover the non-interacting charged KG field. Thus we can view that original version of the charged KG field as being defined by the zero connection. By contrast, in scalar electrodynamics we view the connection as one of the dependent variables of the theory and we therefore have, as we saw, the full gauge invariance since now we can let the connection be transformed along with the KG field.

As you may know from differential geometry, when using a covariant derivative one loses the commutativity of the differentiation process. The commutator of two covariant derivatives defines the *curvature* of the connection. Let us compute this curvature.

$$\left[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu\right] \varphi = iqF_{\mu\nu}\varphi. \tag{70}$$

Thus the Maxwell field strength tensor is the curvature of the covariant derivative! To continue the analogy with differential geometry a bit further, you see that the field φ is playing the role of a vector, with its vector aspect being the fact that it takes values in the vector space $\mathbf{C}$ and transforms homogeneously under the change of fiber coordinates, that is, the gauge transformation. The complex conjugate can be viewed as living in the dual space to C, so that it is a "covector". Quantities like the Lagrangian density, or the conserved electric current are "scalars" from this point of view – they are gauge invariant. In particular, the current

$$j^\mu = -iq(\varphi^* D^\mu \varphi - \varphi \mathcal{D}^\mu \varphi^*) \tag{71}$$

is divergence free with respect to the ordinary derivative, which is the correct covariant derivative on "scalars".

Do we really need all this fancy mathematics? Perhaps not. But, since all the apparatus of gauge symmetry, covariant derivatives *etc.* that show up repeatedly in field theory is naturally arising in this geometric structure, it is clear that this is the right way to be thinking about gauge theories. Moreover, there are certain results that would, I think, be very hard to come by without using the fiber bundle point of view. I have in mind certain important topological structures that can arise via global effects in classical and quantum field theory. These topological structures are, via the physics literature, appearing in the guise of "monopoles" and "instantons". Such structures would play a very nice role in a second semester for this course, if there were one.

PROBLEMS

1. Compute the EL equations of $\mathcal{L}_{KG}$ in (8).

2. Verify the result (30).

3. Derive the formula (37) for j^α.