Electromagnetism: the simplest gauge theory

Electromagnetism

Let us now study some of the salient field theoretic properties of "electromagnetic theory". This is historically the first and also the simplest example of a "gauge theory". We shall see that certain structural features familiar from KG theory appear also for electromagnetic theory and that new structural features appear as well.

We begin with a quick review of Maxwell's equations.

PROBLEM: Maxwell's equations for the electric and magnetic field $(\vec{E}, \vec{B})$ associated to charge density and current density $(\rho, \vec{j})$ are given by

$$\nabla \cdot \vec{E} = 4\pi\rho, \tag{1}$$

$$\nabla \cdot \vec{B} = 0, \tag{2}$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 4\pi\vec{j}, \tag{3}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{4}$$

Show that for any function ϕ and vector field $\vec{A}(\vec{r}, t)$ the electric and magnetic fields defined by

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A} \tag{5}$$

satisfy the second and fourth Maxwell equations shown above.

PROBLEM: Define the anti-symmetric array $F_{\mu\nu}$ in inertial Cartesian coordinates $x^\alpha = (t, x, y, z), \alpha = 0, 1, 2, 3$ via

$$F_{ti} = -E_i, \quad F_{ij} = \epsilon_{ijk}B^k, \quad i, j = 1, 2, 3. \tag{6}$$

Under a change of inertial reference frame corresponding to a boost along the x axis with speed v the electric and magnetic fields change $(\vec{E}, \vec{B}) \to (\vec{E}', \vec{B}')$, where

$$E^{x\prime} = E^x, \quad E^{y\prime} = \gamma(E^y - vB^z), \quad E^{z\prime} = \gamma(E^z + vB^y) \tag{7}$$

$$B^{x\prime} = B^x, \quad B^{y\prime} = \gamma(B^y + vE^z), \quad B^{z\prime} = \gamma(B^z - vB^y). \tag{8}$$

Show that this is equivalent to saying that $F_{\mu\nu}$ are the components of a tensor of type $\binom{0}{2}$.

PROBLEM: Define the 4-current

$$j^\alpha = (\rho, j^i), \quad i = 1, 2, 3. \tag{9}$$

Show that the Maxwell equations take the form

$$F^{\alpha\beta}{}_{,\beta} = 4\pi j^\alpha, \quad F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \tag{10}$$

where indices are raised and lowered with the usual Minkowski metric.

PROBLEM: Show that the scalar and vector potentials, $(\phi, \vec{A})$, when assembled into the 4-potential

$$A_\mu = (\phi, A_i), \quad i = 1, 2, 3, \tag{11}$$

are related to the electromagnetic tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{12}$$

This is the general (local) solution to the homogeneous Maxwell equations $F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$.

While the electromagnetic field can be described solely by the field tensor F in Maxwell's equations, if we wish to use a variational principle to describe this field theory we will have to use potentials.* So, we will describe electromagnetic theory using the scalar and vectr potentials, which can be viewed as a spacetime *1-form*

$$A = A_\alpha(x) dx^\alpha. \tag{13}$$

Depending upon your tastes, you can think of this 1-form as (1) a (smooth) section of the cotangent bundle of the spacetime manifold M; (2) tensor field of type $\binom{0}{1}$; (3) a connection on a $U(1)$ fiber bundle; (4) a collection of 4 functions, $A_\alpha(x)$ defined in a given coordinate system x^α and such that in any other coordinate system $x^{\alpha\prime}$

$$A'_\alpha(x') = \frac{\partial x^\beta}{\partial x^{\alpha\prime}} A_\beta(x(x')). \tag{14}$$

* It can be shown using techniques from the *inverse problem of the calculus of variations* that there is no variational principle for Maxwell's equations built solely from $(\vec{E}, \vec{B})$ and their derivatives.

In any case, A is called the "Maxwell field", the "electromagnetic field", the "electromagnetic potential", the "gauge field", the "4-vector potential" (!), the "U(1) connection", and some other names as well, along with various mixtures of these.

As always, having specified the geometric nature of the field, the field theory is defined by giving a Lagrangian. To define the Lagrangian we introduce the *field strength tensor* F, also known as the "Faraday tensor" or as the "curvature" of the gauge field A. We write

$$F = F_{\alpha\beta}(x) dx^\alpha \otimes dx^\beta. \tag{15}$$

The field strength is in fact a two-form (an anti-symmetric $\binom{0}{2}$ tensor field):

$$F_{\alpha\beta} = -F_{\beta\alpha}, \tag{16}$$

and we can write

$$F = \frac{1}{2} F_{\alpha\beta}(dx^\alpha \otimes dx^\beta - dx^\beta \otimes dx^\alpha) = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta,$$

The field strength is defined as the exterior derivative of the Maxwell field:

$$F = dA, \tag{17}$$

that is,

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}. \tag{18}$$

This guarantees that $dF = 0$, which is equivalent to the homogeneous Maxwell equations $F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$. Thus the only remaining Maxwell equations to be considered are $F^{\alpha\beta}{}_{,\beta} = 4\pi j^\alpha$.

The 6 independent components of F in an inertial Cartesian coordinate chart (t, x, y, z) define the electric and magnetic fields as perceived in that reference frame. Note, however, that all of the definitions given above are in fact valid on an arbitrary spacetime manifold in an arbitrary system of coordinates.

The Lagrangian for electromagnetic theory – on an arbitrary spacetime (M, g) – can be defined by the n-form (where $n = dim(M)$),

$$\mathbf{L} = -\frac{1}{4} F \wedge *F = \mathcal{L}\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \tag{19}$$

where $*F$ is the Hodge dual defined by the spacetime metric g. In terms of components in a coordinate chart we have the Lagrangian density given by

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g} F^{\alpha\beta} F_{\alpha\beta}, \tag{20}$$

where

$$F^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}. \tag{21}$$

Of course, we can – and usually will – restrict attention to the flat spacetime in the standard Cartesian coordinates for explicit computations. It is always understood that F is built from A in what follows.

Let us compute the Euler-Lagrange derivative of $\mathcal{L}$. For simplicity we will work on flat spacetime in inertial Cartesian coordinates so that

$$M = \mathbf{R}^4, \quad g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz \tag{22}$$

We have

$$\begin{aligned} \delta\mathcal{L} &= -\frac{1}{2} F^{\alpha\beta} \delta F_{\alpha\beta} \\ &= -\frac{1}{2} F^{\alpha\beta} (\delta A_{\beta,\alpha} - \delta A_{\alpha,\beta}) \\ &= -F^{\alpha\beta} \delta A_{\beta,\alpha} \\ &= F^{\alpha\beta}{}_{,\alpha} \delta A_\beta + D_\alpha \left(-F^{\alpha\beta} \delta A_\beta \right). \end{aligned} \tag{23}$$

From this identity the Euler-Lagrange expression is given by

$$\mathcal{E}^\beta(\mathcal{L}) = F^{\alpha\beta}{}_{,\alpha}, \tag{24}$$

and the *source-free Maxwell equations are*

$$F^{\alpha\beta}{}_{,\alpha} = 0. \tag{25}$$

There are some equivalent expressions of the field equations that are worth knowing about. First of all, we have that

$$F^{\alpha\beta}{}_{,\alpha} = 0 \Longleftrightarrow g^{\alpha\gamma} F_{\alpha\beta,\gamma} \equiv F_{\alpha\beta,}{}^{\alpha} = 0, \tag{26}$$

so that the field equations can be expressed as

$$g^{\alpha\gamma}(A_{\beta,\alpha\gamma} - A_{\alpha,\beta\gamma}) = 0. \tag{27}$$

We write this using the wave operator $\Box$ (which acts component-wise on the 1-form A) and the operator

$$\operatorname{div} A = g^{\alpha\beta} A_{\alpha,\beta} = A_{\alpha,}{}^{\alpha} = A^\alpha{}_{,\alpha}. \tag{28}$$

via

$$\Box A_\beta - (\operatorname{div} A)_{,\beta} = 0. \tag{29}$$

You can see that this is a modified wave equation.

A more sophisticated expression of the field equations, which is manifestly valid on any spacetime, uses the technology of differential forms. Recall that on a spacetime one has the *Hodge dual*, which identifies the space of p-forms with the space of $n-p$ forms. This mapping is denoted by

$$\alpha \to *\alpha. \tag{30}$$

If F is a 2-form, then $*F$ is an $(n-2)$-form. The field equations are equivalent to the vanishing of a 1-form:

$$*d*F = 0, \tag{31}$$

where d is the exterior derivative. This equation is valid on any spacetime (M, g) and is equivalent to the EL equations for the Maxwell Lagrangian as defined above on any spacetime.

Let me show you how this formula works in our flat 4-d spacetime. The components of $*F$ are given by

$$*F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}. \tag{32}$$

The exterior derivative maps the 2-forms $*F$ to a 3-form $d*F$ via

$$(d*F)_{\alpha\beta\gamma} = 3\partial_{[\alpha} * F_{\beta\gamma]} = \frac{3}{2}\partial_{[\alpha}\epsilon_{\beta\gamma]\mu\nu}F^{\mu\nu}. \tag{33}$$

The Hodge dual maps the 3-form $d*F$ to a 1-form $*d*F$ via

$$\begin{aligned}(*d*F)\sigma &= \frac{1}{6}\epsilon_{\sigma\alpha\beta\gamma}(d*F)^{\alpha\beta\gamma} \\ &= \frac{1}{4}\epsilon_{\sigma\alpha\beta\gamma}\epsilon^{\beta\gamma\mu\nu}\partial^\alpha F_{\mu\nu} \\ &= -\delta_\sigma^{[\mu}\delta_\alpha^{\nu]}\partial^\alpha F_{\mu\nu} \\ &= F_{\alpha\sigma},^\alpha .\end{aligned} \tag{34}$$

So that

$$*d*F = 0 \Longleftrightarrow F_{\alpha\beta},^\alpha = 0. \tag{35}$$

The following two problems are easy to do, but they establish some key structural features of electromagnetic theory.

PROBLEM: Show that the EL derivative of the Maxwell Lagrangian satisfies the differential identity

$$D_\beta\mathcal{E}^\beta(\mathcal{L}) = 0. \tag{36}$$

PROBLEM: Restrict attention to flat spacetime in Cartesian coordinates, as usual. Fix a vector field on spacetime, $j^\alpha = j^\alpha(x)$. Show that the Lagrangian

$$\mathcal{L}_j = -\frac{1}{4}\sqrt{-g}F^{\alpha\beta}F_{\alpha\beta} + j^\alpha A_\alpha \tag{37}$$

gives the field equations

$$F^{\alpha\beta}{}_{,\alpha} = -j^{\beta}. \tag{38}$$

These are the Maxwell equations with prescribed *electric sources* having a charge density ρ and current density $\vec{j}$, where

$$j^{\alpha} = (\rho, \vec{j}). \tag{39}$$

Use the results from the preceding problem to show that the Maxwell equations with sources have no solution unless the vector field representing the sources is divergence-free:

$$\partial_{\alpha} j^{\alpha} = 0. \tag{40}$$

Show that this condition is in fact the usual continuity equations representing conservation of electric charge.

PROBLEM: Show that the Lagrangian density for source-free electromagnetism can be written in terms of the electric and magnetic fields (in any given inertial frame) by $\mathcal{L} = \frac{1}{2}(E^2 - B^2)$. This is one of the 2 relativistic invariants that can be made algebraically from $\vec{E}$ and $\vec{B}$.

PROBLEM: Show that $\vec{E}\cdot\vec{B}$ is relativistically invariant. Express it in terms of potentials and show that it is just a divergence, with vanishing Euler-Lagrange expression.

Gauge symmetry

Probably the most significant aspect of electromagnetic theory, field theoretically speaking, is that it admits an infinite-dimensional group of variational symmetries known as *gauge symmetries*. Their appearance stems from the fact that

$$F = dA, \tag{41}$$

so that if we make a *gauge transformation*

$$A \to A' = A + d\Lambda, \tag{42}$$

where $\Lambda\colon M \to \mathbf{R}$, then

$$F' = dA' = d(A + d\Lambda) = dA = F, \tag{43}$$

where we used the fact that, on any differential form, $d^2 = 0$. It is easy to check all this explicitly:

PROBLEM: For any function $\Lambda = \Lambda(x)$, define

$$A'_{\alpha} = A_{\alpha} + \partial_{\alpha}\Lambda. \tag{44}$$

Show that

$$A'_{\alpha,\beta} - A'_{\beta,\alpha} = A_{\alpha,\beta} - A_{\beta,\alpha}. \tag{45}$$

Show that, in terms of the scalar and vector potentials, this gauge transformation is equivalant to

$$\phi \to \phi' = \phi - \partial_t \Lambda, \quad \vec{A} \to \vec{A}' = \vec{A} + \nabla\Lambda. \tag{46}$$

This shows, then, that under the transformation

$$A \to A' \tag{47}$$

we have

$$F \to F.$$

Evidently, the Lagrangian – which contains A only through F – is invariant under this transformation of A. We say that the Lagrangian is *gauge invariant.*

Mathematically, the gauge transformations are a large set of variational symmetries. Physically, the gauge transformation symmetry has no physical content in the sense that one identifies physical situations described by gauge equivalent Maxwell fields. Thus the Maxwell fields A provide a redundant description of the physics. (Indirectly there is a physical role for this redundancy: the need to use the potentials A can be understood from the desire to have a variational principle (crucial for quantum theory) and the desire to express the theory in a fundamentally *local* form.)

The gauge symmetry is responsible for the fact that the field equations

$$\Box A - d(div\, A) = 0 \tag{48}$$

are not hyperbolic. Indeed, hyperbolic equations will have a Cauchy problem with unique solutions for given initial data. It is clear that, because the function Λ is arbitrary, one can never have unique solutions to the field equations for A associated to given Cauchy data. To see this, let A be any solution for prescribed Cauchy data on a hypersurface $t = const.$ Let A' be any other solution obtained by a gauge transformation:

$$A' = A + d\Lambda. \tag{49}$$

It is easy to see that A' also solves the field equations. This follows from a number of points of view. For example, the field equations are conditions on the field strength F, which is invariant under the gauge transformation. Alternatively, the field equations are invariant under the field equations because the Lagrangian is. Finally, you can check directly that $d\Lambda$ solves the field equations:

$$[(\Box - d\, div)d\Lambda]_\alpha = \partial^\beta \partial_\beta(\partial_\alpha \Lambda) - \partial_\alpha(\partial^\beta \partial_\beta \Lambda) = 0. \tag{50}$$

Since Λ is an arbitrary smooth function, we can choose the first two derivatives of Λ to vanish on the initial hypersurface so that A' and A are distinct solutions with the same initial data.

Noether's second theorem in electromagnetic theory

We have seen that a variational (or divergence) symmetry leads to a conserved current. The gauge transformation defines a variational symmetry for electromagnetic theory. Actually, there are *many* gauge symmetries: because each function on spacetime (modulo an additive constant) defines a gauge transformation, the set of gauge transformations is infinite dimensional! Let us consider our Noether type of analysis for these symmetries. We will see that the analysis that led to Noether's (first) theorem can be taken a little further when the symmetry involves arbitrary functions.

Consider a 1-parameter family of gauge transformations:

$$A' = A + d\Lambda_s, \tag{51}$$

characterized by a 1-parameter family of functions Λ_s where

$$\Lambda_0 = 0. \tag{52}$$

Infinitesimally, we have that

$$\delta A = d\sigma, \tag{53}$$

where

$$\sigma = \left(\frac{\partial \Lambda_s}{\partial s}\right)_{s=0}.$$

It is easy to see that the function σ can be chosen arbitrarily just as we had for field variations in the usual calculus of variations analysis. The Lagrangian is invariant under the gauge transformation; therefore it is invariant under its infinitesimal version. Let us check this explicitly. For any variation we have that

$$\delta\mathcal{L} = -\frac{1}{2}F^{\alpha\beta}\delta F_{\alpha\beta},$$

and under a variation defined by an infinitesimal gauge transformation

$$\begin{aligned}\delta F_{\alpha\beta} &= \partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha\\ &= \partial_\alpha(\partial_\beta\sigma) - \partial_\beta(\partial_\alpha\sigma)\\ &= 0,\end{aligned}$$

so that $\delta\mathcal{L} = 0$.

Now, for any variation the first variational identity is

$$\delta\mathcal{L} = F^{\alpha\beta}{}_{,\alpha}\delta A_\beta + D_\alpha\left(-F^{\alpha\beta}\delta A_\beta\right).$$

For a variation induced by an infinitesimal gauge transformation we therefore must get

$$0 = F^{\alpha\beta}{}_{,\alpha}\partial_\beta\sigma + D_\alpha\left(-F^{\alpha\beta}\partial_\beta\sigma\right), \tag{54}$$

which is valid for any function σ. Now we take account of the fact that the function σ is arbitrary. We can integrate by parts again:

$$0 = -F^{\alpha\beta}{}_{,\alpha\beta}\,\sigma + D_\alpha\left(-F^{\alpha\beta}\partial_\beta\sigma + F^{\alpha\beta}{}_{,\beta}\,\sigma\right) \tag{55}$$

Integrate this identity over a spacetime region $\mathcal{R}$:

$$0 = -\int_{\mathcal{R}} F^{\alpha\beta}{}_{,\alpha\beta}\,\sigma + \int_{\partial\mathcal{R}}\left(-F^{\alpha\beta}\partial_\beta\sigma + F^{\alpha\beta}{}_{,\beta}\,\sigma\right) d\Sigma_\alpha, \tag{56}$$

This must hold for any function σ; we can use the fundamental theorem of variational calculus to conclude that the field equations satisfy the differential identity

$$F^{\alpha\beta}{}_{,\alpha\beta} = 0, \tag{57}$$

which you proved directly in a previous homework problem. Note that this says the Euler-Lagrange expression is divergence-free, and that this holds whether or not the field equations are satisfied – it is an *identity* arising due to the gauge symmetry of the Lagrangian.

We have seen that the gauge symmetry, since it involves arbitrary functions, leads not to a conserved current but rather to an identity satisfied by the field equations. This is an instance of *Noether's second theorem*, and the resulting identity is sometimes called the "Noether identity" associated to the gauge symmetry. Let us have a look at Noether's second theorem in more generality.

Noether's second theorem

We have seen that the identity

$$D_\alpha\mathcal{E}^\alpha(\mathcal{L}) = 0, \tag{58}$$

for

$$\mathcal{E}^\alpha(\mathcal{L}) = F^{\alpha\beta}{}_{,\beta}\,, \tag{59}$$

follows from the gauge invariance of $\mathcal{L}$. Let us give a fairly general statement of this phenomenon, which is a version of *Noether's second theorem.*

Consider a system of fields φ^a, $a = 1, 2, \ldots$, described by a Lagrangian $\mathcal{L}$ and field equations defined by

$$\delta\mathcal{L} = \mathcal{E}_a(\mathcal{L})\delta\varphi^a + D_\alpha\eta^\alpha, \tag{60}$$

where η^α is a linear differential function of $\delta\varphi^a$.

Let us define an *infinitesimal gauge transformation* to be an infinitesimal transformation

$$\delta\varphi^a = \delta\varphi^a(\Lambda), \tag{61}$$

that is defined as a linear differential operator $\mathcal{D}$ on arbitrary functions Λ^A:

$$\delta\varphi^a(\Lambda) = [\mathcal{D}(\Lambda)]^a. \tag{62}$$

The gauge transformation is an *infinitesimal gauge symmetry* if it leaves the Lagrangian invariant up to a divergence of a spacetime vector field $W^\alpha(\Lambda)$ constructed as a linear differential operator on the functions Λ^A, $A = 1, 2, \ldots$:

$$\delta\mathcal{L} = D_\alpha W^\alpha(\Lambda). \tag{63}$$

Noether's second theorem now asserts that the existence of a gauge symmetry implies differential identities satisfied by the field equations. To see this, we simply use the fact that, for any functions Λ^A,

$$0 = \delta\mathcal{L} - D_\alpha W^\alpha = \mathcal{E}_a(\mathcal{L})[\mathcal{D}(\Lambda)]^a + D_\alpha(\eta^\alpha - W^\alpha), \tag{64}$$

where both η and W are linear differential functions of Λ. As before, we integrate this identity over a region and choose the functions Λ^A to vanish in a neighborhood of the boundary so that the divergence terms can be neglected. We then have that, for all functions Λ^A,

$$\int_{\mathcal{R}} \mathcal{E}_a(\mathcal{L})[\mathcal{D}(\Lambda)]^a = 0. \tag{65}$$

Now imagine integrating by parts each term in $D^a(\Lambda)$ so that all derivatives of Λ are removed. The boundary terms that arise vanish. This process defines the *formal adjoint* $\mathcal{D}^*$ of the linear differential operator $\mathcal{D}$:

$$\int_{\mathcal{R}} \mathcal{E}_a\mathcal{D}^a(\Lambda) = \int_{\mathcal{R}} \Lambda^A[\mathcal{D}^*(\mathcal{E})]_A, \tag{66}$$

and we have that

$$\int_{\mathcal{R}} \Lambda^A[\mathcal{D}^*(\mathcal{E})]_A = 0. \tag{67}$$

The fundamental theorem of variational calculus then tells us that the Euler-Lagrange expressions must obey the differential identities:

$$[\mathcal{D}^*(\mathcal{E})]_A = 0. \tag{68}$$

You can easily track this argument through our Maxwell example. The gauge transformation is defined by the differential operator – the exterior derivative on functions –

$$[\mathcal{D}(*)]_\alpha = \partial_\alpha \Lambda. \tag{69}$$

The infinitesimal transformation

$$\delta A_\alpha = [\mathcal{D}(*)]_\alpha \tag{70}$$

is a symmetry of the Lagrangian with

$$W^\alpha = 0. \tag{71}$$

The adjoint of the exterior derivative is given by the divergence:

$$V^\alpha \partial_\alpha \Lambda = -\Lambda \partial_\alpha V^\alpha + \text{ divergence}, \tag{72}$$

so that

$$[\mathcal{D}^*(\mathcal{E})] = \partial_\alpha \mathcal{E}^\alpha, \tag{73}$$

which leads to the Noether identity

$$\partial_\alpha \mathcal{E}^\alpha = 0 \tag{74}$$

for any field equations coming from a gauge invariant Lagrangian.

PROBLEM: Consider the electromagnetic field coupled to sources with the Lagrangian density

$$\mathcal{L}_j = -\frac{1}{4}\sqrt{-g} F^{\alpha\beta} F_{\alpha\beta} + j^\alpha A_\alpha \tag{75}$$

Show that this Lagrangian is gauge invariant if and only if the spacetime vector field j^α is chosen to be divergence-free. What is the Noether identity in this case?

Poincaré symmetry and the canonical energy-momentum tensor for electromagnetic theory

The Maxwell Lagrangian only depends upon the spacetime (M, g) for its construction. Because the Poincaré group is the symmetry group of Minkowski space, we again have the result that, assuming that $M = \mathbf{R}^4$ and g is flat, the Poincaré group acts as a symmetry group. For your convenience I remind you that this group acts on spacetime with inertial Cartesian coordinates x^α via

$$x^\alpha \to x^{\alpha\prime} = M^\alpha_\beta x^\beta + a^\alpha,$$

where a^α are constants defining a spacetime translation and the constant matrix M^α_β defines a Lorentz transformation:

$$M^\alpha_\gamma M^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta},$$

where

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To every electromagnetic potential A_α we have a 10 parameter family of potentials obtained by letting the Poincaré group act via the pull-back operation on 1-forms:

$$A'_\alpha(x) = M^\beta_\alpha A_\beta(M \cdot x + a).$$

You can interpret the transformed potential as describing the electromagnetic field in the transformed reference frame. Because the Lagrangian is the same in all reference frames, these transformations define a 10 parameter family of (divergence) symmetries of the Lagrangian and corresponding conservation laws. Let us focus on the spacetime translation symmetry.

Consider a 1-parameter family of translations, say,

$$a^\alpha = \lambda b^\alpha.$$

We have then

$$\delta A_\alpha = b^\beta A_{\alpha,\beta}. \tag{76}$$

This implies that

$$\delta F_{\mu\nu} = b^\alpha F_{\mu\nu,\alpha} \tag{77}$$

and hence that

$$\delta\mathcal{L} = -\frac{1}{2} b^\gamma F^{\alpha\beta} F_{\alpha\beta,\gamma} = D_\gamma(-\frac{1}{4} b^\gamma F^{\alpha\beta} F_{\alpha\beta}) \tag{78}$$

Recalling the first variational identity:

$$\delta\mathcal{L} = F^{\alpha\beta}{}_{,\alpha} \delta A_\beta + D_\alpha \left(-F^{\alpha\beta}\delta A_\beta\right),$$

this leads to the conserved current

$$j^\alpha = -b^\gamma \left(F^{\alpha\beta} A_{\beta,\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu} F_{\mu\nu}\right). \tag{79}$$

You can easily check with a direct computation that j^α is conserved, that is,

$$D_\alpha j^\alpha = 0, \tag{80}$$

when the field equations hold.

PROBLEM: Verify equations (76)–(80).

Since this must hold for each constant vector b^α, we can summarize these conservation laws using the *canonical energy-momentum tensor*

$$\mathcal{T}^\alpha_\gamma = \left(F^{\alpha\beta}A_{\beta,\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu}F_{\mu\nu}\right),$$

which satisfies

$$D_\alpha \mathcal{T}^\alpha_\beta = 0, \tag{81}$$

modulo the field equations.

There is one glaring defect in the structure of the canonical energy-momentum tensor: it is not gauge invariant. Indeed, under a gauge transformation

$$A \longrightarrow A + d\Lambda \tag{82}$$

we have

$$\mathcal{T}^\alpha_\beta \longrightarrow \mathcal{T}^\alpha_\beta + F^{\alpha\mu}\partial_\mu\partial_\beta\Lambda. \tag{83}$$

In order to see what to do about this, we need to consider some flexibility we have in defining conserved currents. This is our next task.

"Trivial" conservation laws.

Faced with the gauge dependence of the canonical energy momentum tensor of electromagnetic theory, it is now a good time to discuss the notion of *trivial conservation laws*. For any field theory (*e.g.,* KG theory or electromagnetic theory) there are two ways to construct conserved currents that are in some sense "trivial". The first is to suppose that we have a conserved current that actually vanishes when the field equations hold. For example, in the KG theory we could use

$$j^\alpha = (\Box - m^2)\partial^\alpha\varphi. \tag{84}$$

It is, of course, easy to check that this current is conserved. It is even easier to check that this current is completely uninteresting since it vanishes for any solution of the field equations. It is the first type of "trivial" conservation law. Similarly, in electromagnetic theory any vector field proportional to $F^{\alpha\beta}{}_{,\beta}$ is likewise trivial. The triviality of these conservation laws also can be seen by constructing the conserved charge in a region by integrating j^0 over a volume. Of course, when you try to substitute a solution of the equations of motion into j^0 so as to perform the integral you get zero. Thus you end up with the trivial statement that zero is conserved.

The second kind of "trivial" conservation law arises as follows. Suppose we create an anti-symmetric, $\binom{2}{0}$ tensor field locally from the fields and their derivatives:

$$S^{\alpha\beta} = -S^{\beta\alpha}. \tag{85}$$

For example, in KG theory we could use

$$S^{\alpha\beta} = k^\alpha \partial^\beta \varphi - k^\beta \partial^\alpha \varphi, \tag{86}$$

where $k^\alpha = k^\alpha(x)$ is any vector field on spacetime. Now make a current via

$$j^\alpha = D_\beta S^{\alpha\beta}. \tag{87}$$

It is easy to check that such currents are always conserved, irrespective of field equations:

$$D_\alpha j^\alpha = D_\alpha D_\beta S^{\alpha\beta} = 0. \tag{88}$$

These sorts of conservation laws are trivial because they do not really reflect properties of the field equations but rather simple derivative identities analogous to the fact that the divergence of the curl is zero, or that the curl of the gradient is zero.

It is also possible to understand this second kind of trivial conservation laws from the point of view of the conserved charge

$$Q_V = \int_V d^3x\, j^0. \tag{89}$$

For a trivial conservation law arising as the divergence of a skew tensor (locally constructed from the fields) we can integrate by parts, *i.e.,* use the divergence theorem, to express Q_V as a boundary integral:

$$Q_V = \int_S d^2S\, n_i S^{0i}. \tag{90}$$

From the continuity equation, the time rate of change of Q_V arises from the flux through the boundary S of V:

$$\frac{d}{dt} Q_V = -\int_S d^2S\, n_i j^i, \tag{91}$$

where n is the unit normal to the boundary and $i = 1, 2, 3$. But because this continuity equation is an identity (rather than holding by virtue of field equations) this relationship is tautological. To see this, we write:

$$-\int_S d^2S\, n_i j^i = -\int_S d^2S\, n_i \left(S^{i0}{}_{,0} + S^{ij}{}_{,j} \right) = \frac{d}{dt} \int_S d^2S\, n_i S^{0i}. \tag{92}$$

where I used (1) the divergence theorem, and (2) a straightforward application of Stokes theorem in conjunction with the fact that $\partial S = \partial\partial V = \emptyset$ to get

$$\int_S d^2S\, n_i S^{ij}{}_{,j} = 0. \tag{93}$$

Thus the conservation law is really just saying that $\frac{dQ_V}{dt} = \frac{dQ_V}{dt}$ Another way to view this kind of trivial conservation law is to note that the conserved charge is really just a function of the boundary values of the field in the region V and has nothing to do with the state of the field in the interior of V.

To get this last result we used the fact that

$$\int_S d^2S\, n_i S^{ij}{}_{,j} = 0, \tag{94}$$

which is an straightforward application of Stokes theorem in conjunction with the fact that $\partial S = \emptyset$.

PROBLEM: Let S be a two dimensional surface in Euclidean space with unit normal $\vec{n}$ and boundary curve C with tangent $d\vec{l}$. Show that

$$\int_S d^2S\, n_i S^{ij}{}_{,j} = \frac{1}{2}\int_C \vec{V}\cdot d\vec{l}, \tag{95}$$

where

$$V^i = \frac{1}{2}\epsilon^{ijk}S_{jk}.$$

We have seen that there are two kinds of conservation laws that are in some sense trivial. We can of course combine these two kinds of triviality. So, for example, the current

$$j^\alpha = D_\beta(k^{[\beta}\partial^{\alpha]}\varphi) + D^\alpha\varphi(\Box\varphi - m^2\varphi) \tag{96}$$

is trivial.

We can summarize our discussion with a formal definition. We say that a conservation law j^α is *trivial* if there exists a skew-symmetric tensor field $S^{\alpha\beta}$ – locally constructed from the fields and their derivatives – such that

$$j^\alpha = D_\beta S^{\alpha\beta}, \quad \text{modulo the field equations.} \tag{97}$$

Given a conservation law j^α (trivial or non-trivial) we see that we have the possibility to redefine it by adding a trivial conservation law. Thus given one conservation law there are infinitely many others "trivially" related to it. This means that, without some other criterion to choose among these conservation laws, there is no unique notion of "charge density" $\rho = j^0$ since one can change the form of this quantity quite a bit by adding in a trivial conservation law. And, without some specific boundary conditions, there is no unique choice of the *total charge* contained in a region. Usually, there are additional criteria and specific boundary conditions that largely – if not completely – determine the choice of charge density and charge in a region.

Next, let me mention that a nice way to think about trivial conservation laws is in terms of differential forms. On our four dimensional spacetime our vector field j^α can be converted to a 1-form $\omega = \omega_\alpha dx^\alpha$ using the metric:

$$\omega_\alpha = g_{\alpha\beta} j^\beta. \tag{98}$$

This 1-form can be converted to a 3-form $*\omega$ using the Hodge dual

$$(*\omega)_{\alpha\beta\gamma} = g^{\mu\delta} \epsilon_{\alpha\beta\gamma\delta} \omega_\mu. \tag{99}$$

If j is divergence free, this is equivalent to $*\omega$ being *closed*:

$$d(*\omega) = 0, \quad \text{modulo the field equations.} \tag{100}$$

Keep in mind that ω is really a 3-form locally constructed from the field and its derivatives, that is, it is a 3-form valued function on the jet space for the theory. As you know, an *exact* 3-form is of the form $d\beta$ for some 2-form β. If there is a 2-form β locally constructed from the fields such that,

$$\omega = d\beta \quad \text{modulo the field equations} \tag{101}$$

clearly ω is closed. This is just the differential form version of our trivial conservation law. Indeed, the anti-symmetric tensor field that is the "potential" for the conserved current is given by

$$S^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \beta_{\gamma\delta}. \tag{102}$$

Finally, let me mention and dispose of a common point of confusion concerning trivial conservation laws. This point of confusion is why I felt compelled to stick in the phrase "locally constructed from the field" in the discussion above. For simplicity we use the flat metric and Cartesian coordinates on the spacetime manifold $M = \mathbf{R}^4$ in what follows. To set up the difficulty, let me remind you of the following standard result from tensor analysis. Let V^α be a vector field on the manifold M. V^α is not to be viewed as locally constructed from the field, except in the trivial sense that it does not depend upon the fields at all, only the spacetime point, $V^\alpha = V^\alpha(x)$. If V^α is divergence free,

$$\partial_\alpha V^\alpha = 0, \tag{103}$$

then there exists a tensor field $S^{\alpha\beta}$ on M such that

$$V^\alpha = \partial_\beta S^{\alpha\beta}. \tag{104}$$

This is just the dual statement to the well-known fact that all closed 3-forms (indeed, all closed forms of degree higher than 0) on $\mathbf{R}^4$ are exact. Thus, the De Rham cohomology of $\mathbf{R}^4$ is trivial. This kind of result tempts one to conclude that all conservation laws are trivial! Unlike the case in real life, you should not give in to temptation here. Conservation laws should not be viewed as just vector fields on the spacetime manifold M. They are more interesting than that: they are really ways of assigning vector fields on M to solutions to the field equations. In particular, the conservation laws at a point x depend upon the values of the fields and their derivatives at the point x. Thus we say that the conservation laws are locally constructed from the fields, *i.e.,* are functions on jet space (rather than just x space). The correct notion of triviality is that a j^α is trivial if it is (modulo the field equations) a divergence of a skew tensor field $S^{\alpha\beta}$ *that is itself locally constructed from the fields*. If we take a conservation law and evaluate it on a given solution to the field equations, then we end up with a divergence-free vector field on M (or a closed 3-form on M, if you prefer). We can certainly write it as the divergence of a skew tensor on M (or as the exterior derivative of a 2-form on M). But the point is there is no way to express this skew tensor (2-form) as the evaluation on the solution of a local formula in terms of the fields and their derivatives. So, while the set of conservation laws is obtained rather like de Rham cohomology (closed modulo exact forms) it is actually a rather different kind of cohomology. Sometimes this kind of cohomology is called "local cohomology".

New and improved Maxwell energy-momentum tensor

Armed with our new and improved understanding of conservation laws we can revisit the gauge-dependence of the canonical energy-momentum tensor in electromagnetic theory. The canonical energy-momentum tensor is

$$\mathcal{T}^\alpha_\gamma = \left(F^{\alpha\beta}A_{\beta,\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu}F_{\mu\nu}\right),$$

This is not gauge invariant because of the explicit presence of the potentials A. The field strength F *is* gauge invariant and it is possible to show that all local and gauge invariant expressions will depend on the vector potential only through the field strength. With that in mind we can write

$$\begin{aligned}\mathcal{T}^\alpha_\gamma &= \left(F^{\alpha\beta}F_{\beta\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu}F_{\mu\nu}\right) - F^{\alpha\beta}A_{\gamma,\beta}\\ &= \left(F^{\alpha\beta}F_{\beta\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu}F_{\mu\nu}\right) - D_\beta(F^{\alpha\beta}A_\gamma) + A_\gamma D_\beta F^{\alpha\beta}.\end{aligned}$$

You can see that, modulo a set of trivial conservation laws the canonical energy-momentum tensor takes the gauge invariant form

$$T^\alpha_\gamma = F^{\alpha\beta}F_{\beta\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu}F_{\mu\nu}.$$

This tensor, which is equivalent to the canonical energy-momentum tensor modulo trivial conservation laws, is called the "gauge-invariant energy-momentum tensor" or the "improved energy-momentum tensor" or the "general relativistic energy-momentum tensor", since this energy-momentum tensor serves as the source of the gravitational field in general relativity and can be derived using the variational principle of that theory.

The improved energy-momentum tensor has another valuable feature relative to the canonical energy-momentum tensor (besides gauge invariance). The canonical energy-momentum tensor, defined as,

$$\mathcal{T}^\alpha_\gamma = \left(F^{\alpha\beta}A_{\beta,\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu}F_{\mu\nu}\right),$$

is not symmetric. If we define

$$\mathcal{T}_{\alpha\beta} = g_{\alpha\gamma}\mathcal{T}^\gamma_\beta$$

then you can see that

$$\mathcal{T}_{[\alpha\beta]} = F_{[\alpha}{}^\gamma\partial_{\beta]}A_\gamma.$$

Here we used the notation

$$\mathcal{T}_{[\alpha\beta]} \equiv \frac{1}{2}(\mathcal{T}_{\alpha\beta} - \mathcal{T}_{\beta\alpha}).$$

On the other hand the improved energy-momentum tensor,

$$T^\alpha_\gamma = F^{\alpha\beta}F_{\beta\gamma} - \frac{1}{4}\delta^\alpha_\gamma F^{\mu\nu}F_{\mu\nu}.$$

is symmetric:

$$T_{\alpha\beta} = T_{\beta\alpha}.$$

Why is all this important? Well, think back to the KG equation. There, you will recall, the conservation of angular momentum, which stems from the symmetry of the Lagrangian with respect to the Lorentz group, comes from the currents

$$M^{\alpha(\mu)(\nu)} = x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu} = 2T^{\alpha[\mu}x^{\nu]}.$$

These 6 currents were conserved since (1) $T^{\alpha\beta}$ is divergence free (modulo the equations of motion) and (2) $T^{[\alpha\beta]} = 0$. This result will generalize to give conservation of angular momentum in electromagnetic theory using the improved energy-momentum tensor. So the improved tensor in electromagnetic theory plays the same role relative to angular momentum as does the energy-momentum tensor of KG theory.

Why did we have to "improve" the canonical energy-momentum tensor? Indeed, we have a sort of paradox: the Lagrangian is gauge invariant, so why didn't Noether's theorem automatically give us the gauge invariant energy-momentum tensor? As with most paradoxes, the devil is in the details. Noether's theorem involves using the variational identity in the form

$$\delta\mathcal{L} = \mathcal{E}(\mathcal{L}) + D_\alpha \eta^\alpha,$$

to construct the conserved current from η^α (and any divergence which arises in the symmetry transformation of $\mathcal{L}$). To consruct the canonical energy-momentum tensor we used

$$\eta^\alpha = -F^{\alpha\beta}\delta A_\beta, \quad \text{and} \quad \delta A_\beta = b^\gamma A_{\beta,\gamma}$$

to get

$$\eta^\alpha = -b^\gamma F^{\alpha\beta} A_{\beta,\gamma},$$

which we plugged into Noether's theorem. As I have mentioned without explanation here and there via footnotes, there is some ambiguity in the definition of η^α. In light of our definition of trivial conservation laws, I think you can easily see what that ambiguity is. Namely, the variational identity only determines η^α up to addition of the divergence of a skew tensor (locally constructed from the fields and field variations). Our simplest looking choice for η^α using the infinitesimal transformation (76) was not the best because it was not gauge invariant. The improved energy-momentum tensor arises by making a gauge invariant choice for η^α.