Symmetries and Conservation Laws.

Conservation laws

In physics, conservation laws are of undisputed importance. They are the keystone for every fundamental theory of nature. They also provide valuable physical information about the complicated behavior of non-linear dynamical systems. From the mathematical point of view, when analyzing the mathematical structure of differential equations and their solutions the existence of conservation laws (and their attendant symmetries via Noether's theorem) are also very important. We will now spend some time studying some conservation laws for the KG equation. Later we will introduce the notion of symmetry and then describe a version of the famous Noether's theorem relating symmetries and conservation laws. As usual, we begin by defining everything in terms of the example at hand: the KG field theory. It will then not be hard to see how the idea of conservation laws works in general.

We say that a vector field on spacetime, constructed as a local function,

$$j^\alpha = j^\alpha(x, \varphi, \partial\varphi, \dots, \partial^k\varphi) \tag{1}$$

is a *conserved current* or defines a *conservation law* if the divergence of j^α vanishes whenever φ satisfies its field equations (the KG equation). We write

$$D_\alpha j^\alpha = 0, \quad \text{when } (\Box - m^2)\varphi = 0. \tag{2}$$

It is understood that the relations between derivatives defined by the field equations and all the subsequent relations which can be obtained by differentiating them are imposed. Note that we are using the total derivative notation, which is handy when viewing j^α as a function on jet space, that is, as a being built via some function of finitely many variables. The idea of the conservation law is that it provides a formula for taking any given solution the field equations

$$\varphi = \varphi(x), \quad (\Box - m^2)\varphi(x) = 0, \tag{3}$$

and then building a vector field on spacetime – also called j^α by a standard abuse of notation –

$$j^\alpha(x) := j^\alpha(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \dots, \frac{\partial^k\varphi(x)}{\partial x^k}) \tag{4}$$

such that

$$\frac{\partial}{\partial x^\alpha} j^\alpha(x) = 0. \tag{5}$$

You can easily see in inertial Cartesian coordinates that our definition of a conserved current simply says that the field equations imply a continuity equation for the density $\rho(x)$ (a function on spacetime) and the current density $\vec{j}(x)$ (a time dependent vector field on space) associated with any solution $\varphi(x)$ of the field equations:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0, \tag{6}$$

where

$$\rho(x) = j^0(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \dots, \frac{\partial^k\varphi(x)}{\partial x^k}), \tag{7}$$

and

$$(\vec{j})^i = j^i(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \dots, \frac{\partial^k\varphi(x)}{\partial x^k}), \quad i = 1, 2, 3. \tag{8}$$

The utility of the continuity equation is as follows. Define the *total charge contained in the region* V of space at a given time t to be

$$Q_V(t) = \int_V d^3x\, \rho(x). \tag{9}$$

Note that the total charge is a functional of the field, that is, its value depends upon which field you choose. The integral over V of the continuity equation implies that

$$\frac{d}{dt}Q_V(t) = -\int_{\partial V} \vec{j}\cdot\mathbf{dS}\ . \tag{10}$$

Keep in mind that this relation is only valid when the field is a solution to the field equation.

PROBLEM: Derive (10) from the continuity equation.

We call the right hand side of (10) the *net flux into* V. The idea is then that the charge Q_V is conserved since we can account for its time rate of change purely in terms of the flux into or out of the region V. In this sense there is no "creation" or "destruction" of the charge, although the charge can move from place to place.

With suitable boundary conditions, one can choose V large enough such that Q_V contains "all the charge in the universe" for all time. In this case, by definition, charge cannot enter or leave the region and so the total charge is constant in time. In this case we speak of a *constant of the motion.* For example, we have seen that a reasonable set of boundary conditions to put on the KG field (motivated, say, by the variational principle) is to assume that the KG field vanishes at spatial infinity. Let us then consider the region V to be all of space, that is, $V = \mathbf{R}^3$. If the fields vanish at spatial infinity fast enough, then the net flux will vanish and we will have

$$\frac{dQ_V}{dt} = 0. \tag{11}$$

Conservation of energy

Let us look at an example of a conservation law for the KG equation. Consider the spacetime vector field locally built from the KG field and its first derivatives via

$$j^0 = \frac{1}{2}\left(\varphi_{,t}^2 + (\nabla\varphi)^2 + m^2\varphi^2\right), \tag{12}$$

$$j^i = -\varphi_{,t}(\nabla\varphi)^i.$$

Let us see how this defines a conserved current. We compute

$$D_0 j^0 = \varphi_{,t}\varphi_{,tt} + \nabla\varphi\cdot\nabla\varphi_{,t} + m^2\varphi\varphi_{,t}, \tag{13}$$

and

$$D_i j^i = -(\nabla\varphi_{,t})\cdot(\nabla\varphi) - \varphi_{,t}\nabla^2\varphi. \tag{14}$$

All together, we get

$$\begin{aligned} D_\alpha j^\alpha &= \varphi_{,t}\left(\varphi_{,tt} - \nabla^2\varphi + m^2\varphi\right) \\ &= -\varphi_{,t}(\Box\varphi - m^2\varphi). \end{aligned} \tag{15}$$

Obviously, then, if we substitute a solution $\varphi = \varphi(x)$ into this formula, the resulting vector field $j^\alpha(x)$ will be conserved.

The conserved charge Q_V associated with this conservation law is called the *energy of the KG field* in the region V and is denoted by E_V:

$$E_V = \int_V d^3x \frac{1}{2}\left(\varphi_{,t}^2 + (\nabla\varphi)^2 + m^2\varphi^2\right). \tag{16}$$

There are various reasons why we view this as an energy. First of all, if you put in physically appropriate units, you will find that E_V has the dimensions of energy. The best reason comes from Noether's theorem, which we shall discuss later. For now, let us recall that the Lagrangian has the form

$$L = T - U, \tag{17}$$

where the "kinetic energy" is given by

$$T = \int_V d^3x\, \frac{1}{2}\varphi_{,t}^2, \tag{18}$$

and the "potential energy" is given by

$$U = \int_V d^3x \frac{1}{2}\left((\nabla\varphi)^2 + m^2\varphi^2\right). \tag{19}$$

Natually, then, the conserved charge that arises as

$$E_V = T + U \tag{20}$$

is called the total energy (in the region V).

The net flux of energy into V is given by

$$-\int_{\partial V} \vec{j} \cdot \mathbf{dS} = \int_{\partial V} \varphi_{,t} \nabla\varphi \cdot \mathbf{dS}. \tag{21}$$

If we choose $V = \mathbf{R}^3$ then the total energy of the KG field – in the whole universe – is independent of time if the product of the time rate of change of φ and the radial derivative of φ vanish as $r \to \infty$ faster than $\frac{1}{r^2}$. Of course, for the total energy to be defined in this case the integral of the *energy density* $j^0(x)$ must exist and this imposes decay conditions on the solutions $\varphi(x)$ to the KG equation. Indeed, we must have that $\varphi(x)$, its time derivative, and the magnitude of its spatial gradient should decay "at infinity" faster than $\frac{1}{r^{3/2}}$. This guarantees that the net flux into $\mathbf{R}^3$ vanishes.

Conservation of momentum

Let us look at another conservation law for the KG equation known as the *conservation of momentum*. These actually arise as a triplet of conservation laws in which

$$\rho_{(i)} = \varphi_{,t}\varphi_{,i}, \quad i = 1, 2, 3 \tag{22}$$

$$(\vec{j}_{(i)})^l = -(\nabla\varphi)^l\varphi_{,i} + \frac{1}{2}\delta^l_i \left[(\nabla\varphi)^2 - (\varphi_{,t})^2 + m^2\varphi^2\right]. \tag{23}$$

PROBLEM: Verify that the currents

$$j^\alpha_{(i)} := (\rho_{(i)}, \vec{j}_{(i)}) \tag{24}$$

are conserved. (If you like, you can just fix a value for i, say, $i = 1$ and check that j^α_1 is conserved.)

The origin of the name "momentum" of these conservation laws can be understood on the basis of units: the conserved charges

$$P_{(i)} = \int_V d^3x\, \varphi_{,t}\varphi_{,i}, \tag{25}$$

have the dimensions of momentum (if one takes account of the various dimensionful constants that we have set to unity). The name can also be understood from the fact that the each of the three charge densities $\rho_{(i)}$ corresponds to a component of the current densities for the energy conservation law. Roughly speaking, you can think of this quantity as getting the name "momentum" since it defines the "flow of energy". In a little while we will get an even better explanation from Noether's theorem. Finally, recall that the total momentum of a system is represented as a vector in $\mathbf{R}^3$. The components of this vector in the case of a KG field are the $P_{(i)}$.

Energy-momentum tensor

The conservation of energy and conservation of momentum can be given a unified treatment by introducing a $\binom{0}{2}$ tensor field on spacetime known as the *energy-momentum tensor* (also known as the "stress-energy-momentum tensor", the "stress-energy tensor", and the "stress tensor"). Given a KG field $\varphi\colon \mathbf{R}^4 \to \mathbf{R}$ (not necessarily satisfying any field equations), the energy-momentum tensor is defined as

$$T = d\varphi \otimes d\varphi - \frac{1}{2} g^{-1}(d\varphi, d\varphi) g - \frac{1}{2} m^2 \varphi^2 g, \tag{26}$$

where g is the metric tensor of spacetime. Our conservation laws were defined for the KG field on flat spacetime and the formulas were given in inertial Cartesian coordinates $x^\alpha = (t, x^i)$ such that the metric takes the form

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta, \tag{27}$$

with

$$g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1). \tag{28}$$

The formula given for the energy-momentum tensor is in fact correct on any spacetime. The components of the energy-momentum tensor take the form

$$T_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \varphi_{,\gamma}\varphi_{,\delta} - \frac{1}{2} m^2 \varphi^2 g_{\alpha\beta}. \tag{29}$$

This tensor field is symmetric:

$$T_{\alpha\beta} = T_{\beta\alpha}. \tag{30}$$

If desired, one can view this formula as defining a collection of functions on jet space representing a formula for a tensor field on spacetime.

You can check that the conserved energy current is given by

$$j^\alpha_{energy} = -T^\alpha_t \equiv -g^{\alpha\beta} T_{t\beta}. \tag{31}$$

In particular the energy density is T^{tt}. Likewise, the conserved momentum currents are given by

$$j^\alpha_{momentum} = -T^\alpha_i \equiv -g^{\alpha\beta} T_{i\beta}, \quad i = 1, 2, 3,$$

so that, in particular, the momentum density is given by $-T^{ti}$. The conservation of energy and momentum are encoded in the important identity:

$$g^{\beta\gamma} D_\gamma T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi, \tag{32}$$

where we used the fact that

$$\Box\varphi = g^{\alpha\beta}\varphi_{,\alpha\beta}. \tag{33}$$

This relation shows that the energy-momentum tensor is divergence-free when it is evaluated on a field $\varphi = \varphi(x)$ satisfying the KG equation.

Although we are not stressing relativistic considerations in our presentation, it is perhaps worth mentioning that there is no absolute distinction between energy and momentum. A change of reference frame will mix up these quantities. One therefore usually speaks of the "conservation of energy-momentum", or the "conservation of four-momentum" represented by the four currents

$$j^{\beta}_{(\alpha)} = -T^{\beta}_{\alpha} = -g^{\beta\gamma}T_{\alpha\gamma}. \tag{34}$$

Conservation of angular momentum

Finally, we mention 6 more conservation laws known as the conservation laws of *relativistic angular momentum*, which are given by the following 6 currents:

$$M^{\alpha(\mu)(\nu)} = T^{\alpha\mu}x^{\nu} - T^{\alpha\nu}x^{\mu}. \tag{35}$$

Note that

$$M^{\alpha(\mu)(\nu)} = -M^{\alpha(\nu)(\mu)}, \tag{36}$$

which is why there are only 6 independent currents.

PROBLEM: Show that these 6 currents are conserved. (*Hint: Don't panic! This is actually the easiest one to prove so far, since you can use*

$$g^{\beta\gamma}D_{\gamma}T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi, \tag{37}$$

which we have already established.)

In a given inertial reference frame labeled by coordinates $x^{\alpha} = (t, x^i) = (t, x, y, z)$ the relativistic angular momentum naturally decomposes into two pieces in which (α, β) take the values (i, j) and $(0, i)$. Let us look at the charge densities; we have

$$\rho^{(i)(j)} := M^{0(i)(j)} = T^{0i}x^{j} - T^{0j}x^{i}, \tag{38}$$

$$\rho^{(0)(i)} := M^{0(0)(i)} = T^{00}x^{i} - T^{0i}t. \tag{39}$$

The first charge density represents the usual notion of (density of) angular momentum. Indeed, you can see that it has the familiar *position* $\times$ *momentum* form. The second charge density, $\rho^{(0)(i)}$, when integrated over a region V yields a conserved charge which can be interpreted, roughly, as the "center of energy at $t = 0$" in that region. Just as energy and momentum are two facets of a single, relativistic energy-momentum, you can think of these two conserved quantities as forming a single relativistic form of angular momentum.

Let us note that while the energy-momentum conserved currents are (in Cartesian coordinates) local functions of the fields and their first derivatives, the angular momentum conserved currents are also explicit functions of the spacetime events. Thus we see that conservation laws are, in general, functions on the full jet space $(x, \varphi, \partial\varphi, \partial^2\varphi, \ldots)$.

Variational symmetries

Let us now, apparently, change the subject to consider the notion of symmetry in the context of the KG theory. We shall see that this is not really a change in subject when we come to Noether's theorem relating symmetries and conservation laws.

A slogan for the definition of symmetry in the style of the late John Wheeler would be "change without change". When we speak of an object admitting a "symmetry", we usually have in mind some kind of transformation of that object that leaves some specified aspects of that object unchanged. We can partition transformations into two types: *discrete* and *continuous*. The continuous transformations depend continuously on one or more parameters. For example, the group of rotations of $\mathbf{R}^3$ about the z-axis defines a continuous transformation parametrized by the angle of rotation. The "inversion" transformation

$$(x, y, z) \to (-x, -y, -z) \tag{40}$$

is an example of a discrete transformation. We will be focusing primarily on continuous transformations in what follows.

For a field theory such as the KG theory, let us define a *one-parameter family of transformations* – also called a *continuous transformation* – to be a rule for building from any given field φ a family of KG fields (not necessarily satisfying any field equations), denoted by φ_λ. We always assume that the transformation starts at $\lambda = 0$ in the sense that

$$\varphi_{\lambda=0} = \varphi. \tag{41}$$

You are familiar with such "curves in field space" from our discussion of the variational calculus. We also assume that the transformation defines a unique curve through each point in the space of fields. We view this as a transformation of any field φ, that varies continuously with the parameter λ, and such that $\lambda = 0$ is the identity transformation. As an example, we could have a transformation

$$\varphi(x) \longrightarrow \varphi_\lambda(x) := e^\lambda \varphi(x), \tag{42}$$

which is an example of a *scaling* transformation. As another example, we could have

$$\varphi(t,x,y,z) \longrightarrow \varphi_\lambda(t,x,y,z) := \varphi(t+\lambda,x,y,z), \tag{43}$$

which is the transformation induced on φ due to a *time translation*.

We say that a continuous transformation is a *continuous variational symmetry* for the KG theory if it leaves the Lagrangian invariant in the sense that, for any KG field $\varphi = \varphi(x)$,

$$\mathcal{L}(x, \varphi_\lambda(x), \frac{\partial\varphi_\lambda(x)}{\partial x}) = \mathcal{L}(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}). \tag{44}$$

Explicitly, we want

$$-\frac{1}{2}\sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha\varphi_\lambda(x)\partial_\beta\varphi_\lambda(x) + m^2\varphi_\lambda^2\right) = -\frac{1}{2}\sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha\varphi(x)\partial_\beta\varphi(x) + m^2\varphi^2\right). \tag{45}$$

An equivalent way to express this is that

$$\frac{\partial}{\partial\lambda}\mathcal{L}(x, \varphi_\lambda(x), \frac{\partial\varphi_\lambda(x)}{\partial x}) = 0. \tag{46}$$

I think you can see why this is called a "symmetry". While the KG field is certainly changed by a non-trivial symmetry, from the point of view of the Lagrangian nothing is changed by this field transformation.

Our treatment of variational symmetries did not rely in any essential way upon the continuous nature of the transformation. For example, you can easily see that the discrete transformation

$$\varphi \to -\varphi \tag{47}$$

leaves the KG Lagrangian unchanged and so would be called a *discrete variational symmetry*. Any transformation of the KG field that leaves the Lagrangian unchanged will be called simply a *variational symmetry*. Noether's theorem, which is our goal, involves continuous variational symmetries.

Infinitesimal symmetries

Let us restrict our attention to continuous symmetries. A fundamental observation going back to Lie is that, when considering aspects of problems involving continuous transformations, it is always best to formulate the problems in terms of *infinitesimal transformations*. Roughly speaking, the technical advantage provided by an infinitesimal description is that many non-linear equations that arise become linear. The idea of an infinitesimal transformation is that we consider the change in the field that arises for "very small" values of the parameter λ. More precisely, we define the infinitesimal change of the field in much the same way as we do a field variation,

$$\delta\varphi = \left(\frac{\partial\varphi_\lambda}{\partial\lambda}\right)_{\lambda=0}, \tag{48}$$

which justifies the use of the same notation, I think. This notation, which is standard, is also a bit confusing. A field variation in a variational principle involves studying all curves through a specific point (a critical point) in field space so that, for each curve, $\delta\varphi$ is a single function on spacetime. An infinitesimal transformation $\delta\varphi$ will be a spacetime function which will depend upon the field φ being transformed. And it is this dependence which is the principal object of study. From a more geometric point of view, field variations in the calculus of variations represent tangent vectors at a single point in the space of fields. An infinitesimal transformation is a vector *field* on the space of fields – a continuous assignment of a vector to each point in field space.

An example is in order. For the scaling transformation

$$\varphi_\lambda = e^\lambda \varphi, \tag{49}$$

we get

$$\delta\varphi = \varphi, \tag{50}$$

which shows quite clearly that $\delta\varphi$ is built from φ and so it varies from point to point in the space of fields. Likewise for time translations:

$$\varphi_\lambda(t, x, y, z) = \varphi(t + \lambda, x, y, z) \tag{51}$$

$$\delta\varphi = \varphi_{,t}. \tag{52}$$

Of course, just as it is possible to have a constant vector field, it is possible to have a continuous transformation whose infinitesimal form happens to be independent of φ. For example, given some function $f = f(x)$ the transformation

$$\varphi_\lambda = \varphi + \lambda f \tag{53}$$

has the infinitesimal form

$$\delta\varphi = f. \tag{54}$$

This transformation is sometimes called a *field translation*.

The infinitesimal transformation gives a formula for the "first-order" change of the field under the indicated continuous transformation. This first order information is enough to completely characterize the transformation. The idea is that a finite continuous transformation can be viewed as being built by composition of "many" infinitesimal transformations. Indeed, if you think of a continuous transformation as a family of curves foliating field space, then an infinitesimal transformation is the vector field defined by the tangents to those curves at each point. As you may know, it is enough to specify the vector field to determine the foliation of curves (via the "flow of the vector field"). If this bit of mathematics is obscure to you, then you may be happier by recalling that, say, the "electric field lines" are completely determined by specifying the electric vector field.

For a continuous transformation to be a variational symmetry it is necessary and sufficient that its infinitesimal form defines an (infinitesimal) variational symmetry. By this I mean that the variation induced in $\mathcal{L}$ by the infinitesimal transformation vanishes for all fields φ. That this condition is necessary is clear from our earlier observation that a continuous symmetry satisfies can be defined by:

$$\frac{\partial}{\partial\lambda}\mathcal{L}(x, \varphi_\lambda(x), \frac{\partial\varphi_\lambda(x)}{\partial x}) = 0. \tag{55}$$

Clearly, this implies that

$$\begin{aligned} 0 = &= \frac{\partial}{\partial\lambda}\mathcal{L}(x, \varphi_\lambda(x), \frac{\partial\varphi_\lambda(x)}{\partial x})\Big|_{\lambda=0} \\ &= \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\delta\varphi_\alpha \\ &= \delta L. \end{aligned} \tag{56}$$

This last equation is precisely the condition that the infinitesimal change in the Lagrangian induced by the infinitesimal transformation must vanish, so the infinitesimal symmetry condition is necessary. That this condition is also sufficient follows from the fact that it must hold at all points in the space of fields, so that the derivative with respect to λ vanishes everywhere on the space of fields. Thus one often checks whether a continuous transformation is a variational symmetry by just checking its infinitesimal condition (56).

Divergence symmetries

We have defined a (variational) symmetry as a transformation that leaves the Lagrangian invariant. This is a reasonable state of affairs since the Lagrangian determines the field equations (via the Euler-Lagrange equations), and the conservation laws are enforced by the field equations, as we have seen. However, we have also seen that any two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ differing by a divergence

$$\mathcal{L}' = \mathcal{L} + D_\alpha W^\alpha \tag{57}$$

will define the same EL equations since

$$0 = \mathcal{E}(D_\alpha W^\alpha) = \mathcal{E}(\mathcal{L}') - \mathcal{E}(\mathcal{L}). \tag{58}$$

Therefore, it is reasonable to generalize our notion of symmetry ever so slightly. We say that a transformation is a *divergence symmetry* if the Lagrangian only changes by the addition of a divergence. In infinitesimal form, a divergence symmetry satisfies

$$\delta\mathcal{L} = D_\alpha W^\alpha, \tag{59}$$

for some spacetime vector field W^α, built from the KG field. Of course, a variational symmetry is just a special case of a divergence symmetry arising when $W^\alpha = 0$.

You can check that the scaling transformation is neither a variational symmetry nor a divergence symmetry for the KG Lagrangian. On the other hand, the time translation symmetry

$$\delta\varphi = \varphi_{,t} \tag{60}$$

is a divergence symmetry of the KG Lagrangian. Let us show this. We begin by writing this Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\sqrt{|g|}\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + m^2\varphi^2\right), \tag{61}$$

where $g^{\alpha\beta} = \text{diag}(-1,1,1,1)$. We then have

$$\begin{aligned}\delta\mathcal{L} &= -\sqrt{|g|}\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta t} + m^2\varphi\varphi_{,t}\right)\\ &= D_t\mathcal{L}\\ &= D_\alpha\left(\delta^\alpha_t\mathcal{L}\right),\end{aligned} \tag{62}$$

so that we can choose

$$W^\alpha = \delta^\alpha_t\mathcal{L}. \tag{63}$$

Physically, the presence of this symmetry reflects the fact that there is no preferred instant of time in the KG theory. A shift in the origin of time $t \to t+$ *constant* does not change the field equations.

A first look at Noether's theorem

We now have enough technology to have a first, somewhat informal look at Noether's theorem relating symmetries and conservation laws. The idea is as follows. Consider a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi). \tag{64}$$

Of course, the KG Lagrangian is of this form. Suppose that $\delta\varphi$ is an infinitesimal variational symmetry. Then,

$$\delta\mathcal{L} = 0 \tag{65}$$

everywhere in the space of fields But, at any given point in field space, we can view the transformation $\delta\varphi$ as a field variation, so that we can take advantage of the identity we used to compute the EL equations:

$$\delta\mathcal{L} = \mathcal{E}(\mathcal{L})\delta\varphi + D_\alpha V^\alpha, \tag{66}$$

where

$$\mathcal{E}(\mathcal{L}) = \frac{\partial\mathcal{L}}{\partial\varphi} - D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\right), \tag{67}$$

and*

$$V^\alpha = \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} \delta\varphi. \tag{68}$$

This identity holds for *any field variation.* By hypothesis, our field variation is some field built from φ that has the property that $\delta\mathcal{L} = 0$, so that we have the relation

$$D_\alpha V^\alpha = -\mathcal{E}(\mathcal{L})\delta\varphi. \tag{69}$$

This is exactly the type of identity that defines a conserved current V^α since it says that the divergence of V^α will vanish if V^α is built from a KG field φ that satisfies the EL-equation (the KG equation). Note that the specific form of V^α as a function of φ (and its derivatives) depends upon the specific form of the Lagrangian via $\frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}}$ and on the specific form of the transformation via $\delta\varphi$.

More generally, suppose that the infinitesimal transformation $\delta\mathcal{L}$ defines a divergence symmetry, that is, there exists a vector field W^α built from φ such that

$$\delta\mathcal{L} = D_\alpha W^\alpha. \tag{70}$$

We still get a conservation law since our variational identity becomes

$$D_\alpha W^\alpha = \mathcal{E}(\mathcal{L})\delta\varphi + D_\alpha V^\alpha, \tag{71}$$

which implies

$$D_\alpha(V^\alpha - W^\alpha) = -\mathcal{E}(\mathcal{L})\delta\varphi. \tag{72}$$

To summarize, if $\delta\varphi$ is a divergence symmetry of $\mathcal{L}(x, \varphi, \partial\varphi)$,

$$\delta\mathcal{L} = D_\alpha W^\alpha, \tag{73}$$

then there is a conserved current given by

$$j^\alpha = \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} \delta\varphi - W^\alpha. \tag{74}$$

Time translation symmetry and conservation of energy

We can now easily see how the conserved current associated with conservation of energy arises via the time translation symmetry. Recall that time translation symmetry is a divergence symmetry:

$$\delta\varphi = \varphi_{,t} \Longrightarrow \delta\mathcal{L} = D_\alpha(\delta^\alpha_t \mathcal{L}). \tag{75}$$

* There is an ambiguity in the definition of V^α here which we shall ignore for now to keep things simple. We will confront it when we study conservation laws in electromagnetism.

We can therefore apply our introductory version of Noether's theorem to obtain a conserved current:

$$\begin{aligned} j^\alpha &= -g^{\alpha\beta}\varphi_{,\beta}\varphi_{,t} - \delta^\alpha_t \mathcal{L} \\ &= -T^\alpha_t, \end{aligned} \tag{76}$$

which is our expression of the conserved energy current in terms of the energy-momentum tensor.

By the way, it is not hard to see that the existence of the time translation symmetry, and hence conservation of energy, is solely due to the fact that the KG Lagrangian has no explicit t dependence. Consider any Lagrangian whatsover

$$\mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi, \ldots) \tag{77}$$

satisfying

$$\frac{\partial}{\partial t}\mathcal{L}(x, \varphi, \partial\varphi, \ldots) = 0 \qquad (78 = 0.)$$

From the identity

$$D_t\mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \varphi}\varphi_{,t} + \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}}\varphi_{,t\alpha}$$

we have

$$\frac{\partial \mathcal{L}}{\partial \varphi}\varphi_{,t} + \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}}\varphi_{,t\alpha} = D_t\mathcal{L},$$

so that the time translation $\delta\varphi = \varphi_{,t}$ will yield a divergence symmetry and hence conservation of energy. One says that the conserved current for energy is the *Noether current* associated to time translational symmetry.

Space translation symmetry and conservation of momentum

We can also use spatial translation symmetry to obtain conservation of momentum. The symmetry is defined, for $i = 1, 2, 3$, by

$$\varphi(t, x^i) \to \varphi_\lambda(t, x^i) = \varphi(t, x^i + \lambda b^i), \tag{79}$$

where b^i are three constants – a constant vector field $\mathbf{b}$ in space. Infinitesimally, we have

$$\delta\varphi = \mathbf{b}\cdot\nabla\varphi = b^i\varphi_{,i}. \tag{80}$$

To check that this is a symmetry of the KG Lagrangian we compute

$$\begin{aligned} \delta\mathcal{L} &= \varphi_{,t}(\mathbf{b}\cdot\nabla\varphi_{,t}) - \nabla\varphi\cdot\nabla(\mathbf{b}\cdot\nabla\varphi) - m^2\varphi(\mathbf{b}\cdot\nabla\varphi) \\ &= \mathbf{b}\cdot\nabla\mathcal{L} \\ &= \nabla\cdot(\mathbf{b}\mathcal{L}) \\ &= D_\alpha W^\alpha, \end{aligned} \tag{81}$$

where

$$W^\alpha = (0, b^i \mathcal{L}). \tag{82}$$

As before, it is not hard to see that this result is a sole consequence of the fact that the Lagrangian has no dependence on the spatial coordinates. In particular, we have

$$b^i \frac{\partial}{\partial x^i} \mathcal{L}(x, \varphi, \partial\varphi, \ldots) = 0. \tag{83}$$

Thus we have the conservation law

$$j^\alpha = (\rho, j^i), \tag{84}$$

with

$$\rho = \varphi_{,t} \mathbf{b} \cdot \nabla\varphi, \tag{85}$$

and

$$j^i = -\varphi_{,i} \mathbf{b} \cdot \nabla\varphi + \frac{1}{2} b^i \left((\nabla\varphi)^2 - \varphi_{,t}^2 + m^2\varphi^2 \right). \tag{86}$$

Since the vector $\mathbf{b}$ is arbitrary, it is easy to see that we really have three independent conservation laws here corresponding to 3 linearly independent choices for $\mathbf{b}$. These three conservation laws correspond to the conservation laws for momentum that we had before. The relation between ρ and j^i here and $\rho_{(i)}$ and $\vec{j}_{(i)}$ there is given by

$$\rho = b^k \rho_{(k)}, \quad j^i = b^k (\vec{j}_{(k)})^i. \tag{87}$$

You can see that the translational symmetry in the spatial direction defined by $\mathbf{b}$ leads to a conservation law for the component of momentum along $\mathbf{b}$. Thus the three conserved momentum currents are the Noether currents associated with spatial translation symmetry.

Conservation of energy-momentum. Energy-momentum tensor.

We can treat the conservation of energy-momentum in a nice, unified fashion using our four-dimensional tensor notation. Let a^α be any constant vector field on spacetime. Consider the continuous transformation, a *spacetime translation*,

$$\varphi_\lambda(x^\alpha) = \varphi(x^\alpha + \lambda a^\alpha), \quad \delta\varphi = a^\alpha \varphi_{,\alpha}. \tag{88}$$

As a nice exercise you should check that we then have

$$\delta\mathcal{L} = D_\alpha(a^\alpha \mathcal{L}) \tag{89}$$

so that from Noether's theorem we have that

$$j^\alpha = -\sqrt{|g|} g^{\alpha\beta} \varphi_{,\beta} (a^\gamma \varphi_{,\gamma}) - a^\alpha \mathcal{L} \tag{90}$$

is conserved. By choosing a^α to define a time or space translation we get the corresponding conservation of energy or momentum.

Since a^α are arbitrary constants, it is easy to see that for each value of γ, the current

$$j^\alpha_{(\gamma)} = -\sqrt{|g|}g^{\alpha\beta}\varphi_{,\beta}\varphi_{,\gamma} - \delta^\alpha_\gamma \mathcal{L} \tag{91}$$

is conserved, corresponding to the four independent conservation laws of energy and momentum. Substituting for the KG Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\sqrt{|g|}\left(g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2}m^2\varphi^2\right), \tag{92}$$

we get that

$$j^\alpha_{(\gamma)} = -\sqrt{|g|}T^\alpha_\gamma \equiv -\sqrt{|g|}g^{\alpha\beta}T_{\beta\gamma}. \tag{93}$$

Thus the energy-momentum tensor can be viewed as set of Noether currents associated with spacetime translational symmetry.

PROBLEM: Verify that the Noether currents associated with a spacetime translation do yield the energy-momentum tensor.

Angular momentum revisited

We have seen the correspondence between spacetime translation symmetry and conservation of energy-momentum. What symmetry is responsible for conservation of angular momentum? It is *Lorentz symmetry.* Recall that the Lorentz group is a combination of "boosts" and spatial rotations. By definition, a Lorentz transformation is a linear transformation on the spacetime $\mathbf{R}^4$,

$$x^\alpha \longrightarrow S^\alpha_\beta x^\beta, \tag{94}$$

that leaves invariant the quadratic form

$$g_{\alpha\beta}x^\alpha x^\beta = -t^2 + x^2 + y^2 + z^2. \tag{95}$$

We have then

$$S^\alpha_\gamma S^\beta_\delta g_{\alpha\beta} = g_{\gamma\delta}. \tag{96}$$

Consider a 1-parameter family of such transformations, $S(\lambda)$, such that

$$S^\alpha_\beta(0) = \delta^\alpha_\beta, \quad \left(\frac{\partial S^\alpha_\beta}{\partial\lambda}\right)_{\lambda=0} =: \omega^\alpha_\beta \tag{97}$$

Infinitesimally we have that

$$\omega^\alpha_\gamma g_{\alpha\delta} + \omega^\beta_\delta g_{\gamma\beta} = 0. \tag{98}$$

Defining

$$\omega_{\alpha\beta} = g_{\beta\gamma}\omega^\gamma_\alpha \tag{99}$$

we see that a Lorentz transformation is "generated" by ω if and only if the array $\omega_{\alpha\beta}$ is anti-symmetric:

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}. \tag{100}$$

Consider the following transformation

$$\varphi_\lambda(x^\alpha) = \varphi(S^\alpha_\beta(\lambda)x^\beta), \tag{101}$$

so that infinitesimally we have

$$\delta\varphi = (\omega^\alpha_\beta x^\beta)\varphi_{,\alpha}, \tag{102}$$

with an antisymmetric $\omega_{\alpha\beta}$ as above. It is now a short computation to check that, for the KG Lagrangian,

$$\delta\mathcal{L} = D_\alpha\left(\omega^\alpha_\beta x^\beta \mathcal{L}\right). \tag{103}$$

This relation follows from

$$\omega^\alpha_\gamma g_{\alpha\delta} + \omega^\beta_\delta g_{\gamma\beta} = 0. \tag{104}$$

The resulting Noether current is given by

$$\begin{aligned} j^\alpha &= -g^{\alpha\beta}\varphi_{,\beta}(\omega^\gamma_\delta x^\delta)\varphi_{,\gamma} - \omega^\alpha_\beta x^\beta \mathcal{L} \\ &= \omega_{\gamma\delta}M^{\alpha(\gamma)(\delta)}, \end{aligned} \tag{105}$$

where $M^{\alpha(\gamma)(\delta)}$ are the conserved currents associated with relativistic angular momentum.

Spacetime symmetries

The symmetry transformations that we have been studying involve spacetime translations:

$$x^\alpha \longrightarrow x^\alpha + \lambda a^\alpha, \tag{106}$$

where $a^\alpha = const.$ and Lorentz transformations,

$$x^\alpha \longrightarrow S^\alpha_\beta(\lambda)x^\beta, \tag{107}$$

where

$$S^\alpha_\beta S^\gamma_\delta \eta_{\alpha\gamma} = \eta_{\beta\delta}. \tag{108}$$

These symmetries are, naturally enough, called *spacetime symmetries* since they involve transformations in spacetime. These symmetry transformations have a nice geometric interpretation which goes as follows.

Given a spacetime (M, g) we can consider the group of diffeomorphisms, which are smooth mappings of M to itself with smooth inverses. Given a diffeomorphism

$$f: M \to M, \tag{109}$$

there is associated to the metric g a new metric f^*g via the pull-back. In coordinates x^α on M the diffeomorphism f is given as

$$x^\alpha \to f^\alpha(x), \tag{110}$$

and the pullback metric has components related to the components of g via

$$(f^*g)_{\alpha\beta}(x) = \frac{\partial f^\gamma}{\partial x^\alpha}\frac{\partial f^\delta}{\partial x^\beta} g_{\gamma\delta}(f(x)). \tag{111}$$

We say that f is an *isometry* if

$$f^*g = g. \tag{112}$$

The idea of an isometry is that it is a symmetry of the metric – the spacetime points have been moved around, but the metric can't tell it happened. Consider a 1-parameter family of diffeomorphisms f_λ such that $f_0 = \mathit{identity}$. It is not hard to see that the tangent vectors at each point of the flow of f_λ defines a vector field X. The *Lie derivative* of the metric along X is defined as

$$L_X g := \left(\frac{d}{d\lambda} f^*_\lambda g\right)_{\lambda=0}. \tag{113}$$

If f_λ is a 1-parameter family of isometries then we have that

$$L_X g = 0. \tag{114}$$

It is not too hard to verify that the spacetime translations and the Lorentz translations define isometries of the Minkowski metric

$$g = \eta_{\alpha\beta} dx^\alpha \otimes dx^\beta. \tag{115}$$

In fact, it can be shown that all continuous isometries of flat spacetime are contained in the *Poincaré group*, which is the group of diffeomorphisms built from spacetime translations and Lorentz transformations.

The KG Lagrangian depends upon a choice of spacetime for its definition. Recall that a spacetime involves specifying two structures: a manifold M and a metric g on

M. Isometries are symmetries of that structure: they are diffeomorphisms – symmetries of M – that also preserve the metric. It is not too surprising then that the Lagrangian symmetries that we have been studying are symmetries of the spacetime since that is the only structure that is used to construct the Lagrangian. The existence of conservation laws of energy, momentum and angular momentum is contingent upon the existence of suitable spacetime symmetries.

Internal symmetries

There is another class of symmetries in field theory that is very important since, for example, it is the source of other conservation laws besides energy, momentum and angular momentum. This class of symmetries is known as the *internal symmetries* since they do not involve any transformations in spacetime, but only on the space of fields.

There is an easy to spot discrete internal symmetry for the KG theory. You can easily check that the transformation $\varphi \to -\varphi$ does not change the Lagrangian. This symmetry extends to self-interacting KG theories with potentials which are an even function of φ, *e.g.,* the double well potential. There are no interesting continuous internal symmetries of the KG theory unless one sets the rest mass to zero. Then we have the following situation.

PROBLEM: Consider the KG theory with $m = 0$. Show that the transformation

$$\varphi_\lambda = \varphi + \lambda \tag{116}$$

is a variational symmetry. Use Noether's theorem to find the conserved current and conserved charge.

The charged KG field and its internal symmetry

I will let you play with that simple example and move on to a slightly new field theory that admits an internal symmetry, the *charged KG field*. The charged KG field can be viewed as a mapping

$$\varphi: M \to \mathbf{C}, \tag{117}$$

so that there are really two real-valued functions in this theory. The Lagrangian for the charged KG field is

$$\mathcal{L} = -\sqrt{-g}(g^{\alpha\beta}\varphi_{,\alpha}\varphi^*_{,\beta} + m^2|\varphi|^2). \tag{118}$$

PROBLEM: Show that this Lagrangian is the sum of the Lagrangians for two (real-valued) KG fields φ_1 and φ_2 with $m_1 = m_2$ and with the identification

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2).$$

From this problem you can surmise that the field equations for the charged KG field consist of two identical KG equations for the real and imaginary parts of φ. In terms of the complex-valued function φ you can check that the field equations – computed as Euler-Lagrange equations or via the critical points of the action – are simply

$$\mathcal{E}_\varphi(\mathcal{L}) = (\Box - m^2)\varphi^* = 0, \tag{119}$$

$$\mathcal{E}_{\varphi^*}(\mathcal{L}) = (\Box - m^2)\varphi = 0. \tag{120}$$

Note that one can do our field-theoretic computations using the real functions φ_1 and φ_2 or using the familiar trick of using "complex coordinates" on the space of fields, that is, treating φ and φ^* as independent variables. In any case, one has doubled the size of the field space. As we shall see, the new "degrees of freedom" that have been introduced allow for a notion of conserved electric charge. Additionally, in the corresponding quantum field theory they also allow for the introduction of "anti-particles".

It is easy to see that the Lagrangian for the charged KG field admits the symmetry transformation

$$\varphi_\lambda = e^{i\lambda}\varphi, \quad \varphi^*_\lambda = e^{-i\lambda}\varphi^* \tag{121}$$

This continuous variational symmetry is given various names. Sometimes it is called a "phase transformation", sometimes a "rigid $U(1)$ transformation", sometimes a "gauge transformation of the first kind", sometimes a "global $U(1)$ transformation", and sometimes various mixtures of these terms. Whatever the name, you can see that it is simply a rotation in the space of values of the fields φ_1 and φ_2 that were defined in the last problem. The Lagrangian is rotationally invariant in field space, hence the symmetry.

It is straightforward to compute the conserved current associated with the $U(1)$ symmetry, using Noether's theorem. The only novel feature here is that we have more than one field. We therefore give the gory details. The infinitesimal transformation is given by

$$\delta\varphi = i\varphi, \quad \delta\varphi^* = -i\varphi^* \tag{122}$$

The variation of the Lagrangian is, in general, given by

$$\delta\mathcal{L} = \mathcal{E}_\varphi(\mathcal{L})\delta\varphi + \mathcal{E}_{\varphi^*}(\mathcal{L})\delta\varphi^* + D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi^*_{,\alpha}}\delta\varphi^*\right). \tag{123}$$

From the phase symmetry we know that when we set $\delta\varphi = i\lambda\varphi$ it follows that $\delta\mathcal{L} = 0$, so we have

$$0 = \mathcal{E}_\varphi(\mathcal{L})i\lambda\varphi - \mathcal{E}_{\varphi^*}(\mathcal{L})i\lambda\varphi^* + D_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}i\lambda\varphi - \frac{\partial\mathcal{L}}{\partial\varphi^*_{,\alpha}}i\lambda\varphi^*\right). \tag{124}$$

Using

$$\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}} = g^{\alpha\beta}\varphi^*_{,\beta}, \tag{125}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi^*_{,\alpha}} = g^{\alpha\beta}\varphi_{,\beta}, \tag{126}$$

we get a conserved current

$$j^\alpha = -ig^{\alpha\beta}\left(\varphi^*\varphi_{,\beta} - \varphi\varphi^*_{,\beta}\right). \tag{127}$$

PROBLEM: Verify directly from the above formula for j^α that

$$D_\alpha j^\alpha = 0, \tag{128}$$

when the field equations for φ and φ^* are satisfied.

The total "$U(1)$ charge" contained in a spatial volume V at $t = const.$ is given by

$$Q = \int_V d^3x \left(\varphi^*\varphi_{,t} - \varphi\varphi^*_{,t}\right). \tag{129}$$

Note that the sign of this charge is indefinite: the charged KG field contains both positive and negative charges. This charge can be used to model electric charge in electrodynamics. It can also be used to model the charge which interacts via neutral currents in electroweak theory.

More generally. . .

We can generalize our previous discussion as follows. Recall that, given a group G, a (linear) representation is a pair (r, V) where V is a vector space and $r: G \to GL(V)$ is a group homomorphism, that is, r is an identification of linear transformations $r(g)$ on V with elements $g \in G$ such that

$$r(g_1g_2) = r(g_1)r(g_2). \tag{130}$$

This way of viewing things applies to the $U(1)$ symmetric charged Klein-Gordon theory as follows. For the charged KG field the group $G = U(1)$, the set of phases $e^{i\lambda}$ labeled by λ with group multiplication being ordinary multiplication of complex numbers. The vector space was $V = \mathbf{C}$, and the representation r was via multiplication of elements $z \in \mathbf{C}$ by the phase $z \to r(\lambda)z = e^{i\lambda}z$. In that case, the internal symmetry arose as a transformation of the complex-valued field, which makes sense since we can view φ as a map from spacetime into the representation vector space, in this case, $\mathbf{C}$ where the group acts according to its representation.

The generalization of the $U(1)$ symmetric charged KG field to a general group is now clear. Given a group G one picks a representation (r, V). One considers fields that are maps into V, we write

$$\varphi: M \to V. \tag{131}$$

Each element $g \in G$ defines a field transformation via

$$\varphi \longrightarrow \rho(g)\varphi. \tag{132}$$

While it is possible to generalize still further, this construction captures almost all instances of (finite-dimensional) internal symmetries in field theory. Of course, for the transformation just described to be a (divergence) symmetry, it is necessary that the Lagrangian be suitably invariant under the action of $r(g)$. One can examine this issue quite generally, but we will be content with exhibiting another important example.

SU(2) symmetry

The group $SU(2)$ can be defined as the group of unitary, unimodular transformations of the vector space $\mathbf{C}^2$, equipped with its standard inner-product and volume element.* In terms of the Hermitian conjugate (complex-conjugate-transpose) †, the unitarity condition on a linear transformation U is

$$U^\dagger = U^{-1}, \tag{133}$$

which is equivalent to saying that the linear transformation preserves the standard Hermitian scalar product. The unimodularity condition is

$$\det U = 1, \tag{134}$$

which is equivalent to saying that the linear transformation preserves the standard volume form on C^2.

Let us focus on the "defining representation" of $SU(2)$ as just stated. Then the representation vector space is again $\mathbf{C}^2$ and each element of $SU(2)$ can be represented by a matrix of the form

$$r(g) = U(\theta, \mathbf{n}) = \cos\theta\, I + i\sin\theta\, n^i\sigma_i, \tag{135}$$

where

$$\mathbf{n} = (n^1, n^2, n^3), \quad (n^1)^2 + (n^2)^2 + (n^3)^2 = 1, \tag{136}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{137}$$

* This way of defining $SU(2)$ in terms of a representation provides the "defining representation".

are the *Pauli matrices*. Note that there are three free parameters in this group, corresponding to θ and two free parameters defining n^i.*

We can use this group representation to define a transformation group of a field theory using the general strategy we outlined earlier. The fields are defined to be mappings

$$\varphi: M \to \mathbf{C}^2, \tag{138}$$

so we now have two charged KG fields or, equivalently, four real KG fields. You can think of φ as a 2-component column vector whose entries are complex functions on spacetime. Let $U(\lambda)$ be any one parameter family of $SU(2)$ transformations, as described above. We assume that

$$U(0) = I. \tag{139}$$

We define

$$\varphi_\lambda = U(\lambda)\varphi. \tag{140}$$

The infinitesimal form of this transformation is

$$\delta\varphi = i\tau\varphi, \tag{141}$$

where τ is a Hermitian, traceless 2×2 matrix defined by

$$\tau = \frac{1}{i}\left(\frac{dU}{d\lambda}\right)_{\lambda=0}. \tag{142}$$

Note that

$$\delta\varphi^\dagger = -i\varphi^\dagger\tau^\dagger = -i\varphi^\dagger\tau. \tag{143}$$

By the way, you can see that τ is traceless and Hermitian by considering our formula for $U(\theta, n)$ above, or by simply noting that $U(\lambda)$ satisfies

$$U^\dagger(\lambda)U(\lambda) = I, \quad \det(U(\lambda)) = 1 \tag{144}$$

for all values of λ. Differentiation of each of these relations and evaluation at $\lambda = 0$ yields the Hermitian ($\tau^\dagger = \tau$) and trace-free conditions, respectively. It is not hard to see that every Hermitian tracefree matrix is a linear combination of the Pauli matrices:

$$\tau = a^i\sigma_i, \tag{145}$$

where $a^{i*} = a^i$. Thus the $SU(2)$ transformations can also be parametrized by the three numbers a^i.

* The elements of $SU(2)$ are parametrized by a unit vector and an angle, just as are elements of the rotation group $SO(3)$. This can be understood in terms of the *spinor representation* of the group of rotations.

On $\mathbf{C}^2$ we have the following Hermitian inner product that is left invariant by the $SU(2)$ transformation:

$$(\varphi_1, \varphi_2) = \varphi_1^\dagger \varphi_2, \tag{146}$$

$$\begin{aligned}(U\varphi_1, U\varphi_2) &= (U\varphi_1)^\dagger (U\varphi_2) \\ &= \varphi_1^\dagger U^\dagger U \varphi_2 \\ &= \varphi_1^\dagger \varphi_2 \\ &= (\varphi_1, \varphi_2).\end{aligned} \tag{147}$$

This allows us to build a Lagrangian

$$\mathcal{L} = -\sqrt{-g}\left[g^{\alpha\beta}(\varphi_{,\alpha}, \varphi_{,\beta}) + m^2(\varphi, \varphi)\right] \tag{148}$$

that has the $SU(2)$ transformation as an internal variational symmetry. Of course, this Lagrangian just describes a pair of charged KG fields (or a quartet of real KG fields). To see this, we write

$$\varphi = \begin{pmatrix}\varphi^1 \\ \varphi^2\end{pmatrix}, \tag{149}$$

and then

$$\mathcal{L} = -\sqrt{-g}\left[g^{\alpha\beta}(\varphi^{1*}_{,\alpha}\varphi^1_{,\beta} + \varphi^{2*}_{,\alpha}\varphi^2_{,\beta}) + m^2(\varphi^{1*}\varphi^1 + \varphi^{2*}\varphi^2)\right]. \tag{150}$$

Representing the components of φ as φ^a, we have the Euler-Lagrange equations

$$\mathcal{E}_a = (\Box - m^2)\varphi^a = 0, \tag{151}$$

which are equivalent to

$$\mathcal{E} = (\Box - m^2)\varphi = 0, \tag{152}$$

in our matrix notation. Of course, the complex (or Hermitian) conjugates of these equations are also field equations.

Just as before, we can use Noether's theorem to find the current that is conserved by virtue of the $SU(2)$ symmetry.

PROBLEM:

Show that for the symmetry $\delta\varphi = i\tau\varphi$ the associated conserved current is given by

$$j^\alpha = i g^{\alpha\beta}(\varphi^\dagger_{,\beta}\tau\varphi - \varphi^\dagger\tau\varphi_{,\beta}). \tag{153}$$

Note that there are three independent conserved currents corresponding to the three independent symmetry transformations. The 3 conserved charges associated with the $SU(2)$ symmetry are usually called "isospin" for historical reasons.

A general version of Noether's theorem

Let us briefly indicate, without proof, a rather general version of Noether's theorem. (This is sometimes called "Noether's first theorem".) Given all of our examples, this theorem should not be very hard to understand.

Consider a field theory described by a set of functions φ^a, $a = 1, 2, \ldots, m$ and a Lagrangian

$$\mathcal{L} = \mathcal{L}(x, \varphi^a, \partial\varphi^a, \ldots, \partial^k\varphi^a), \tag{154}$$

such that the Euler-Lagrange equations arise via the identity

$$\delta\mathcal{L} = \mathcal{E}_a\delta\varphi^a + D_\alpha\eta^\alpha(\delta\varphi), \tag{155}$$

where $\eta^\alpha(\delta\varphi)$ is a linear differential operator on $\delta\varphi^a$ constructed from the fields φ^a and their derivatives via the usual integration by parts procedure.* Suppose that there is an infinitesimal transformation,

$$\delta\varphi^b = F^b(x, \varphi^a, \partial\varphi^a, \ldots, \partial^l\varphi^a) \tag{156}$$

that is a divergence symmetry:

$$\delta\mathcal{L} = D_\alpha W^\alpha, \tag{157}$$

for some W^α locally constructed from x, φ^a, $\varphi^a_{,\alpha}$, *etc.* Then the following is a conserved current:

$$j^\alpha = \eta^\alpha(F) - W^\alpha. \tag{158}$$

Noether's theorem, as it is conventionally stated – more or less as above, shows that symmetries of the Lagrangian beget conservation laws. But the scope of this theorem is actually significantly larger. It is possible to prove a sort of converse to the result shown above to the effect that conservation laws for a system of Euler-Lagrange equations necessarily arise from symmetries of the Lagrangian. It is even possible to prove theorems that assert in a certain sense a one-to-one correspondence between conservation laws and symmetries of the Lagrangian for a wide class of field theories (including the KG field and its variants that have been discussed up until now). There is even more than this! But it is time to move on....

PROBLEMS

1. Derive (10) from the continuity equation.

* There is an ambiguity in the definition of η^α here which we shall ignore for now to keep things simple. We will confront it when we study conservation laws in electromagnetism.

2. Verify that the currents

$$j^{\alpha}_{(i)} := (\rho_{(i)}, \vec{j}_{(i)}) \tag{159}$$

are conserved. (If you like, you can just fix a value for i, say, $i = 1$ and check that j^{α}_1 is conserved.)

3. Show that these 6 currents are conserved. (*Hint: Don't panic! This is actually the easiest one to prove so far, since you can use*

$$g^{\beta\gamma} D_\gamma T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi, \tag{160}$$

which we have already established.)

4. Verify that the Noether currents associated with a spacetime translation do yield the energy-momentum tensor.

5. Consider the KG theory with $m = 0$. Show that the transformation

$$\varphi_\lambda = \varphi + \lambda \tag{161}$$

is a variational symmetry. Use Noether's theorem to find the conserved current and conserved charge.

6. Show that this Lagrangian is the sum of the Lagrangians for two (real-valued) KG fields φ_1 and φ_2 with $m_1 = m_2$ and with the identification

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2).$$

7. Verify directly from the above formula for j^α that

$$D_\alpha j^\alpha = 0, \tag{162}$$

when the field equations for φ and φ^* are satisfied.

8. Show that for the symmetry $\delta\varphi = i\tau\varphi$ the associated conserved current is given by

$$j^\alpha = i g^{\alpha\beta}(\varphi^\dagger_{,\beta}\tau\varphi - \varphi^\dagger\tau\varphi_{,\beta}). \tag{163}$$

9. Let $\hat{\varphi} = F(\varphi)$ be a variational symmetry of a Lagrangian. Show that it maps solutions of the Euler-Lagrange equations to new solutions, that is, if φ is a solution, so is $\hat{\varphi}$. If you like, you can restrict your attentions to Lagrangians $\mathcal{L}(x, \varphi, \partial\varphi)$.

10. As an illustration of the previous result, consider the real KG field with the double well self-interaction potential. Show that $\hat{\varphi} = -\varphi$ is a symmetry. Consider the 3 constant solutions you found in a previous homework problem and check that the symmetry maps solutions to solutions.

11. The double well potential can be generalized to the $U(1)$ case by choosing $V(\varphi) = -a^2|\varphi|^2+b^2|\varphi|^4$. Check that the Lagrangian with this self-interaction potential still admits the $U(1)$ symmetry. (Hint: this is really easy.) Plot the graph of V as a function of the real and imaginary parts of φ. (Hint: you should see why this is often called the "Mexican hat potential" for an appropriate choice for m and a.) Find all solutions of the field equations of the form $\varphi = constant$. How do these solutions transform under the $U(1)$ symmetry?

12. Verify the field equations, symmetries, and conservation laws discussed in this section using the *DifferentialGeometry* package in *Maple*. You will want to read the Help files for the sub-package "JetCalculus", particularly those pertaining to the commands `EulerLagrange` and `Noether`.