Introduction. What is a classical field theory?

Introduction

This course was created to provide information that can be used in a variety of places in theoretical physics, principally in quantum field theory, particle physics, electromagnetic theory, fluid mechanics and general relativity. Indeed, it is usually in such courses that the techniques, tools, and results we shall develop here are introduced – if only in bits and pieces. As explained below, it seems better to give a unified, systematic development of the tools of classical field theory in one place. If you want to be a theoretical/mathematical physicist you must see this stuff at least once. If you are just interested in getting a deeper look at some fundamental/foundational physics and applied mathematics ideas, this is a good place to do it.

The most familiar classical^{*} field theories in physics are probably (i) the "Newtonian theory of gravity", based upon the Poisson equation for the gravitational potential and Newton's laws, and (ii) electromagnetic theory, based upon Maxwell's equations and the Lorentz force law. Both of these field theories are exhibited below. These theories are, as you know, pretty well worked over in their own right. Both of these field theories appear in introductory physics courses as well as in upper level courses. Einstein provided us with another important classical field theory -a relativistic gravitational theory -via his general theory of relativity. This subject takes some investment in geometrical technology to adequately explain. It does, however, also typically get a course of its own. So, these courses are traditionally used to cover a lot of the classical field theory which is needed in a typical core physics curriculum. The other field theories that are important (e.q.,Dirac, Yang-Mills, Klein-Gordon) typically arise, physically speaking, not as classical field theories but as *quantum* field theories, and it is usually in a course in quantum field theory that these other field theories are described. So, it is through such courses that a student gets exposed to the tools and results of classical field theory in a typical physics curriculum. Let me finish off this introduction with a long-winded attempt at trying to explain why I think that a class such as the one we are now beginning provides a good – if not better – way to learn all this material.

This organization of material just described, while natural in some ways, overlooks the fact that field theories have many fundamental features in common – features which are

^{*} Here, and in all that follows, the term "classical" is to mean "not quantum", *e.g.*, as in "the classical limit". Sometimes people use "classical" to also mean non-relativistic; we shall definitely not being doing that here. Indeed, virtually every field theory we shall consider is "relativistic".

most easily understood in the classical limit – and one can get a really good feel for what is going on in "the big picture" by exploring these features in a general, systematic way. Indeed, many of the basic structures appearing in classical field theory (Lagrangians, field equations, symmetries, conservation laws, gauge transformations, *etc.*) are of interest in their own right, and one should try to master them in general, not just in the context of a particular example (e.g., electromagnetism), and not just in passing as one studies the quantum field. Moreover, once one has mastered a sufficiently large set of such field theoretic tools, one is in a good position to discern how this or that field theory differs from its cousins in some key structural way. This serves to highlight physical and mathematical ingredients that make each theory special.

From a somewhat more pragmatic point of view, let me point out that most quantum field theory texts rely quite heavily upon one's facility with the techniques of classical field theory. This is, in part, because many quantum mechanical structures have analogs in a classical approximation to the theory. By understanding the "lay of the land" in the classical theory through a course such as this one, one gets a lot of insight into the associated quantum field theories. It is hard enough to learn quantum field theory without having to also assimilate at the same time concepts that are already present in the much simpler setting of classical field theory. So, if you are hoping to learn quantum field theory some day, this class should help out quite a bit.

A final motivation for the creation and teaching of this course is to support the activities of a number of faculty and students at USU. The geometric underpinnings of classical field theory feature in a wide variety of research projects here. If you want to find out what is going on in this research – or even participate – you need to speak the language. This course is designed to help.

What is a classical field theory?

In physical terms, a classical field is a dynamical system with an infinite number of degrees of freedom. In mathematical terms, a classical field is a physical system described by PDEs whose solutions involve arbitrary functions. Of course, there is much more to the story, but this at least a good first pass at a definition. By contrast, a *mechanical system* is a dynamical system with a finite number of degrees of freedom, and is described by ODEs whose solutions depend upon a finite dimensional space of parameters.

One place where classical fields naturally occur in physics is when non-rigid extended bodies such as bodies of water, elastic solids, strings under tension, and so forth, are described using a classical (as opposed to quantum) continuum approximation to their structure. Another place where classical fields arise – and this place is where we will focus most of our attention – is in a more or less fundamental description of matter and its interactions. Here the ultimate description is via quantum field theory, but the classical approximation sometimes has widespread validity (*e.g.*, Maxwell theory) or the classical approximation can be very useful for understanding the structure of the theory. Here are some simple examples of classical fields in action, just to whet your appetite. Some of these theories will be explored in detail later.

Example: waves in an elastic medium

Let us denote by $u(\mathbf{r}, t)$ the displacement of some observable characteristic of an elastic medium at the position \mathbf{r} at time t from its equilibrium value. For example, if the medium is the air surrounding you, u could represent its compression/rarefaction relative to some standard value. For small displacements, the medium can often be well-modeled by supposing that the *displacement field* u satisfies the *wave equation*

$$\frac{1}{c^2}u_{,tt} - \nabla^2 u = 0, \tag{1.1}$$

where the comma notation indicates partial derivatives, ∇^2 is the Laplacian, and c is a parameter representing the speed of sound in the medium. We say that u is the *field* variable and that the wave equation is the *field equation*.

Example: Newtonian gravitational field

This is the original field theory. Here the field variable is a function $\phi(\mathbf{r}, t)$, the gravitational potential. The gravitational force $\mathbf{F}(\mathbf{r}, t)$ exerted on a (test) mass m at spacetime position (\mathbf{r}, t) is given by

$$\mathbf{F}(\mathbf{r},t) = -m\nabla\phi(\mathbf{r},t). \tag{1.2}$$

The gravitational potential is determined by the mass distribution that is present via the field equation

$$\nabla^2 \phi = -4\pi G\rho,\tag{1.3}$$

where $\rho = \rho(\mathbf{r}, t)$ is the mass density function and G is Newton's constant. Equation (1.3) is just Poisson's equation, of course.

Equations (1.2) and (1.3) embody a pattern in nature which persists even in the most sophisticated physical theories. The idea is as follows. Matter interacts via fields. The fields are produced by the matter, and the matter is acted upon by the fields. In the present case, the gravitational field, represented by the "scalar field" ϕ , determines the motion of mass via (1.2). Conversely, mass determines the gravitational field via (1.3). The next example also exhibits this fundamental pattern of nature.

Example: Maxwell's equations

Here the electromagnetic interaction of particles is mediated by a pair of vector fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, the electric and magnetic fields. The force on a test particle with electric charge q at spacetime event (\mathbf{r}, t) is given by

$$\mathbf{F}(\mathbf{r},t) = q\mathbf{E}(\mathbf{r},t) + \frac{q}{c} \left[\mathbf{v}(t) \times \mathbf{B}(\mathbf{r},t) \right], \qquad (1.4)$$

where $\mathbf{v}(t)$ is the particle's velocity at time t and c is the speed of light in vacuum.^{*} The electromagnetic field is determined from the charge distribution that is present via the Maxwell equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho,\tag{1.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.6}$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j},\tag{1.7}$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.8}$$

where ρ is the charge density and **j** is the electric current density.

As we shall see later, the electromagnetic field is also fruitfully described using potentials. They are defined via

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$
 (1.9)

In each of these examples, the field variable(s) are u, ϕ , (**E**, **B**) – or (ϕ , **A**), and they are determined by PDEs. The "infinite number of degrees of freedom idea" is that, roughly speaking, the general solution to the field equations (wave, Poisson, or Maxwell) involves arbitrary functions. Thus the space of solutions to the PDEs – physically, the set of field configurations permitted by the laws of physics – is infinite dimensional.

^{*} We are using Gaussian units for the electromagnetic field.