Lecture 28

Relevant sections in text: §3.7, 3.9

Total angular momentum (cont.)

To determine the linear combinations of the $m = 0$ product vectors $|+-\rangle$ and $|-+\rangle$ that yield $\mathbf{S}^2$ eigenvectors we use the angular momentum ladder operators:

$$\mathbf{S}_\pm = S_x \pm iS_y = S_{1\pm} + S_{2\pm}.$$

If we apply $\mathbf{S}_-$ to the eigenvector

$$|s = 1, m = 1\rangle = |++\rangle,$$

we get (exercise)

$$|s = 1, m = 0\rangle = \mathbf{S}_-|s = 1, m = 1\rangle = \mathbf{S}_-|++\rangle = (S_{1-} + S_{2-})|++\rangle = \frac{1}{\sqrt{2}}(|-+\rangle + |+-\rangle).$$

The other eigenket $|0, 0\rangle$ must be orthogonal to this vector as well as to the other eigenkets, $|++\rangle$ and $|--\rangle$, from which its formula follows (exercise). All together, we find the *total* angular momentum eigenvectors, $|s, m\rangle$, are related to the *individual* angular momentum (product) eigenkets by:

$$|1, 1\rangle = |++\rangle$$
$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle),$$
$$|1, -1\rangle = |--\rangle,$$
$$|0, 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).$$

These vectors form an orthonormal basis for the Hilbert space, so they are all the linearly independent eigenvectors of $\mathbf{S}^2$ and $\mathbf{S}_z$. The eigenstates with $s = 1$ are called the *triplet* states and the eigenstate with $s = 0$ is the *singlet state*. Notice that by combining two systems with half-angular momentum we end up with a system that allows integer angular momentum only.

A lengthier – but more straightforward – derivation of the eigenvectors $|s, m\rangle$ arises by simply writing the 4×4 matrix for $\mathbf{S}^2$ in the basis of product vectors $|\pm\pm\rangle$, and solving its eigenvalue problem. This is a good exercise. To get this matrix, you use the formula

$$\mathbf{S}^2 = \mathbf{S}_z^2 + +\hbar\mathbf{S}_z + \mathbf{S}_+\mathbf{S}_-.$$

It is straightforward to deduce the matrix elements of this expression among the product states since each of the operators has a simple action on those vectors.

Notice that the states of definite *total* angular momentum, $|s, m\rangle$, are not all the same as the states of definite individual angular momentum, say, $|\pm\pm\rangle$. This is because the total angular momentum is not compatible with the individual angular momentum. For example,

$$[\mathbf{S}^2, S_{1i}] = [S_1^2 + S_2^2 + 2(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}), S_{1i}] \neq 0.$$

A couple of complete sets of commuting observables are given by $(S_1^2, S_2^2, S_{1z}, S_{2z}))$ and $(S_1^2, S_2^2, \mathbf{S}^2, \mathbf{S}_z)$. The eigenvectors of the first set are the product basis $|\pm\pm\rangle$, representing states in which each individual spin angular momentum state is known with certainty. The eigenvectors of the second set are given by $|s, m\rangle$, representing states in which the total angular momentum is known with certainty.

Angular momentum addition in general

We can generalize the above discussion as follows. Suppose we are given two angular momenta $\vec{J}_1$ and $\vec{J}_2$ (*e.g.,* two spins, or a spin and an orbital angular momentum). We can discuss both angular momenta at once using the direct product space as before, with a product basis $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$. We represent the operators on product vectors as

$$\vec{J}_1(|\alpha\rangle \otimes |\beta\rangle) = (\vec{J}|\alpha\rangle) \otimes |\beta\rangle,$$

and

$$\vec{J}_2(|\alpha\rangle \otimes |\beta\rangle) = |\alpha\rangle \otimes (\vec{J}|\beta\rangle),$$

and extend to general vectors by linearity. The product basis $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ is the basis corresponding to the commuting observables provided by $(J_1^2, J_2^2, J_{1z}, J_{2z}))$.

The total angular momentum is defined by

$$\vec{\mathbf{J}} = \vec{J}_1 + \vec{J}_2.$$

A set of commuting observables that includes the total angular momentum is provided by the operators (J_1^2, J_2^2, J^2, J_z). The basis of eigenvectors defined by these observables are denoted $|j_1, J_2, j, m\rangle$. For given values of j_1 and j_2, it can be shown that

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 - 1, j_1 + j_2$$

with (as usual)

$$m = -j, -j + 1, \dots j - 1, j.$$

These two sets of commuting observables are not all compatible, so the set of total angular momentum eigenvectors will be distinct from the eigenvectors of the individual angular momenta.

Spin correlations and quantum weirdness: The EPR argument

The fact that the total spin magnitude was not compatible with the individual spin observables leads to some somewhat dramatic consequences, from a classical physics point of view. This drama was already noted by Einstein-Podolsky and Rosen (EPR) in a famous critique of the completeness of quantum mechanics. Much later, Bell showed that the basic "classical locality" assumption of EPR must be violated, implying in effect that nature is truly as weird as quantum mechanics makes it out to be. Here we give a brief discussion of some of these ideas.

The original EPR idea did not deal with spins, but with a pair of spinless particles. We shall in a moment, following Bohm, deal with a spin system. But it is worth first describing the EPR argument, which goes as follows. Consider two spinless particles characterized by positions x_1, x_2 and momenta p_1, p_2. It is easy to see that the relative position

$$x = x_1 - x_2$$

and the total momentum

$$p = p_1 + p_2$$

commute, so there is a basis of states in which one can specify these observables with arbitrary accuracy. Suppose that an observer measures the position of particle 1 in such a state. Assuming that particles 1 and 2 are well separated, there is no way this experiment on particle 1 can possibly affect particle 2. (This is essentially the EPR locality idea.) Then one has determined particle 2's position with arbitrary accuracy – without disturbing particle two – since by hypothesis, the state of the system was one in which x was known with arbitrary accuracy. Thus there is a complete set of states in which the position of particle 2 is, in effect, known with certainty. Alternatively, one could arrange to measure p_1. By locality, the value of p_2 is undisturbed and is determined with arbitrary accuracy. Thus this same basis of states allows for p_2 to be determined with certainty. One concludes that, given the locality principle, it should be possible to have a complete set of states in which a particle's position and momentum can be known with arbitrary accuracy. But, of course, quantum mechanics allows no such states. Thus either quantum mechanics is incomplete as a theory (unable to give all available information about things like position and momentum), or the theory is non-local in some sense because the locality idea was used to argue that a measurement on particle 1 has no effect on the outcome of measurements on particle 2. The loss of this type of locality was deemed unpalatable by EPR, and so this thought experiment was used to argue against the completeness of quantum mechanics. In fact, the correct conclusion is that quantum mechanics is, in a certain sense, non-local.