

*Physics 6010, Fall 2016*

*Normal Modes. Simultaneous Diagonalization of Quadratic Forms. Forced Oscillations.*

*Relevant Sections in Text: §6.1–6.3, 6.5*

### Oscillations of systems with more than one degree of freedom.

So far we have studied (small) oscillations of systems with a single degree of freedom. Systems with more degrees of freedom can exhibit much more intricate behavior in the vicinity of stable equilibrium. However, this more intricate behavior can always be viewed as the superposition of harmonic motions of decoupled degrees of freedom — the *normal modes* of vibration. We shall first spend a little time developing the general theory, and then we shall spend some time on examples.

Suppose we have a system with  $n$  generalized coordinates  $q^i$ ,  $i = 1, 2, \dots, n$ , and a Lagrangian of the form

$$L = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j - V(q).$$

Recall that we use the summation convention; there is a double sum in the first term of  $L$ . The “metric”  $g_{ij}(q)$  is a *symmetric* array which may depend upon the configuration coordinates. There is no loss of generality by assuming the metric to be symmetric,

$$g_{ij} = g_{ji},$$

since only the symmetric combination appears in the sum over  $i$  and  $j$  (exercise). Usually, the metric is diagonal (*e.g.*, in spherical polar coordinates), but we have seen examples where the metric is off diagonal (*e.g.*, pendulum with moving point of support).

Critical points of  $V$ , *i.e.*, points  $q_0^i$  such that

$$\left(\frac{\partial V}{\partial q^i}\right)(q_0) = 0,$$

define equilibrium configurations of the system (exercise). We will sometimes suppose that the equilibrium is stable, *i.e.*,  $q_0^i$  is a (local) minimum of  $V$ . But you should always be thinking about the other types of equilibrium as we go.

Let us approximate the motion in the neighborhood of a point of equilibrium by defining

$$x^i = q^i - q_0^i,$$

and expanding the Lagrangian in a Taylor series about  $x^i = 0$ . To the first non-trivial order we get (exercise)

$$L \approx \frac{1}{2}M_{ij}\dot{x}^i\dot{x}^j - \frac{1}{2}K_{ij}x^ix^j$$

where

$$M_{ij} = g_{ij}(q_0), \quad K_{ij} = \left( \frac{\partial^2 V}{\partial q^i \partial q^j} \right) (q_0),$$

and we have dropped an irrelevant additive constant  $V(q_0)$ , *i.e.*, we have adjusted the zero of potential energy to be at  $q_0^i$ . We have that  $M_{ij} = M_{ji}$  and we assume that the potential energy is sufficiently smooth so that the matrix of second partial derivatives is symmetric at the critical point  $q_0$ :

$$K_{ij} = K_{ji}.$$

The (approximate) EL equations are (exercise)

$$M_{ij}\ddot{x}^j + K_{ij}x^j = 0,$$

which are coupled system of  $n$  homogeneous, linear ODEs with constant coefficients. Defining  $\vec{x}$  as a column vector with entries  $x^i$ , and viewing  $M_{ij}$  and  $K_{ij}$  as (symmetric) matrices  $M$  and  $K$ , we can write the EL equations in the matrix form (exercise):

$$M\ddot{\vec{x}} = -K\vec{x}.$$

Let us note that if  $q_0^i$  is a point of stable equilibrium then the symmetric matrix  $K$  is positive definite, that is, it can have only positive eigenvalues.\* This is because a negative or zero eigenvalue will correspond to displacements  $x^i$  which either lower or do not change the potential energy in an arbitrarily small neighborhood of the equilibrium point (exercise), which contradicts our assumption of stable equilibrium. Conversely, since every symmetric matrix can be diagonalized, if the eigenvalues are all positive definite then the point  $q_0^i$  is a minimum. Put differently,  $q_0^i$  is a point of stable equilibrium if and only if the *quadratic form*

$$K(\vec{v}) := K_{ij}v^i v^j$$

is *positive definite*, which means  $K(\vec{v}) > 0$  for all  $\vec{v} \neq 0$ . Physically this means that any displacement  $x^i$  from equilibrium will increase the potential energy. All this discussion is just restating standard results from multivariable calculus.

Likewise, positivity of the kinetic energy implies that in any physical application the symmetric matrix  $M$  should be positive definite. This means the quadratic form

$$M(x) := M_{ij}x^i x^j$$

is positive definite, *i.e.*,  $M(\vec{x}) > 0$  for all  $\vec{x} \neq 0$ . We assume this in what follows.

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\* Note that a symmetric, real matrix always admits complete set of eigenvectors with real eigenvalues.

Here's our strategy for analyzing the (approximate) EL equations. We can write the EL equations as (exercise)

$$\ddot{\vec{x}} + (M^{-1}K)\vec{x} = 0,$$

where  $M^{-1}$  exists because  $M$  is positive definite (exercise). Suppose we can find an eigenvector  $\vec{a}$  of  $M^{-1}K$  with eigenvalue  $\omega^2$ . (For now, we allow  $\omega$  to be complex.) Then it is easy to see that we will get a solution the form

$$\mathbf{x} = \vec{a} \cos(\omega t + \beta).$$

Thus, provided  $\omega^2 > 0$ , *i.e.*,  $\omega$  is real and non-vanishing, some combination of the displacements of the system (determined by the eigenvector) is behaving as a harmonic oscillator with frequency  $\omega$ . If we can find enough eigenvectors, we will be able to build the general solution to the EL equations by superposition. Because the general solution should involve  $2n$  integration constants — corresponding to initial positions and initial velocities, we will need  $n$  independent solutions of the above form (exercise), *i.e.*,  $n$  linearly independent eigenvectors, to get the general solution in this manner.

Our goal, then, is to determine the eigenvalues  $\omega^2$  and the eigenvectors  $\vec{a}$ . We will write our eigenvalue equation in the equivalent form (exercise):

$$(K - \omega^2 M)\vec{a} = 0.$$

Thus  $\vec{a}$  is an eigenvector of the matrix  $(K - \omega^2 M)$  with eigenvalue zero. You will recall the basic result from linear algebra that the square matrix  $K - \omega^2 M$  has an eigenvector with eigenvalue zero if and only if (exercise)

$$\det(K - \omega^2 M) = 0.$$

This is the *characteristic equation* for  $\omega$ ; it is a polynomial equation for  $\omega$  of order  $2n$ . Thus  $\omega$ , called the *characteristic frequency*, arises as a root of a polynomial of order  $2n$ .

In general there are  $2n$  roots of the characteristic equation. But notice that if  $\omega$  is a solution, then so is  $-\omega$  since they both yield the same  $\omega^2$ . Since it is  $\omega^2$  which determines the eigenvector, we see that changing the sign of  $\omega$  does not give a new solution (linearly independent eigenvector). Thus, without loss of generality, we can assume that  $\omega > 0$ . Thus we get  $n$  characteristic frequencies. At this point, as far as we know, these frequencies may be non-vanishing and real, or zero, or imaginary. The real solutions correspond to stable directions in configuration space. The imaginary solutions correspond to directions in configuration space relative to which the equilibrium is unstable. The vanishing frequencies correspond to directions in configuration space relative to which we have neutral equilibrium. Corresponding to each of these roots is an eigenvector  $\vec{a}$  and hence a solution  $\vec{x}$  to the EL equations.

It is not hard to see that if the potential energy quadratic form is positive definite then  $\omega^2 > 0$ , so that there are  $n$  real, *positive* characteristic frequencies. To see this, simply note that for any displacement  $\vec{x}$  solving the equations of motion we have

$$\omega^2 M(\vec{x}) = K(\vec{x}).$$

Because both quadratic forms are positive definite,  $M(\vec{x}) > 0$  and  $K(\vec{x}) > 0$ , it follows that

$$\omega^2 > 0,$$

as desired.

Physically speaking, the existence of  $n$  positive roots for  $\omega$  stems from our assumption that  $q_0$  is a point of stable equilibrium, which mathematically means  $K(\vec{x})$  is positive definite. If  $K(\vec{x})$  could be negative or zero, allowing for complex or vanishing frequencies, this would lead to exponential or linear (rather than oscillatory) solutions. Henceforth we assume we have stable equilibrium and just focus on the sinusoidal solutions. You should have no problem adapting our discussion to the other cases (and we will have an example of neutral equilibrium in a little while)

Once we have a characteristic frequency  $\omega$ , we can reconstruct the corresponding  $a^i$  by solving the equation  $(K - \omega^2 M)\vec{a} = 0$ . The solution is guaranteed to exist because  $\omega$  solves the characteristic equation. In this way we have, in fact, found an eigenvector ( $\vec{a}$ ) of  $M^{-1}K$  with eigenvalue  $\omega^2$  (exercise). It is, in general, possible to find  $n$  roots of the characteristic equation along with  $n$  *orthogonal* — and hence linearly independent — eigenvectors (more on this shortly). Each of the (real) solutions obtained through this procedure evolves in time harmonically at the characteristic frequency; these solutions are called *normal modes*. Let us denote the characteristic frequencies by  $\omega_\alpha$  and the corresponding normalized eigenvectors by  $\vec{a}_\alpha = \{a_\alpha^i\}$ ,  $\alpha = 1, 2, \dots, n$ . The general solution to the EL equations is then a superposition of the normal modes:

$$x^k(t) = \sum_{\alpha} c_{\alpha} a_{\alpha}^k \cos(\omega_{\alpha} t + \beta_{\alpha}),$$

where  $c_{\alpha}$  and  $\beta_{\alpha}$  are constants (of integration) determined by initial conditions.

## Normal Modes - The Recipe

Let us summarize the construction of the normal modes of vibration described by the Lagrangian, which approximates the dynamics of a system near equilibrium:

$$L = \frac{1}{2} M_{ij} \dot{x}^i \dot{x}^j - \frac{1}{2} K_{ij} x^i x^j, \quad i, j = 1, \dots, n.$$

First we solve the characteristic equation

$$\det(\omega^2 M - K) = 0$$

for the characteristic frequencies  $\omega_\alpha$ ,  $\alpha = 1, \dots, n$ .

We then solve the linear equations

$$(\omega_\alpha^2 M - K)\vec{a}_\alpha = 0$$

for the corresponding vectors  $\vec{a}_\alpha$ ,  $\alpha = 1, \dots, n$ . The normal modes of vibration are\*

$$\vec{\Theta}_\alpha(t) = \vec{a}_\alpha \cos(\omega_\alpha t + \beta_\alpha).$$

The general motion of the system in the vicinity of stable equilibrium (in the harmonic approximation) is a superposition of the normal modes. The superposition goes over the amplitudes and phases of each oscillator. These coefficients are determined by initial conditions. We have

$$\vec{x}(t) = \sum_{\alpha=1}^n C_\alpha \vec{\Theta}_\alpha(t).$$

### An Elementary Example

As a very simple example of finding normal modes and characteristic frequencies, let us consider a system described by the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega_0^2(x^2 + y^2) - \frac{1}{2}m\alpha^2(x - y)^2.$$

This system can be viewed as two identical one-dimensional harmonic oscillators (natural frequency  $\omega_0$ ) with a coupling by a harmonic force (natural frequency  $\alpha$ ) (exercise).

The matrices  $M$  and  $K$  are given by (good exercise!)

$$M_{ij} = m\delta_{ij},$$

and

$$K_{xx} = K_{yy} = m(\omega_0^2 + \alpha^2), \quad K_{xy} = K_{yx} = -m\alpha^2.$$

We now have

$$K - \omega^2 M = m \begin{pmatrix} \alpha^2 + \omega_0^2 - \omega^2 & -\alpha^2 \\ -\alpha^2 & \alpha^2 + \omega_0^2 - \omega^2 \end{pmatrix}.$$

\* Here we assume  $\omega_\alpha \neq 0$ . If a characteristic frequency vanishes the corresponding normal mode is of the form  $\vec{\Theta} = (c + dt)\vec{a}$ , where  $c$  and  $d$  are constants.

The characteristic equation is (exercise)

$$0 = \det(K - \omega^2 M) = m^2 \left[ (\alpha^2 + \omega_0^2 - \omega^2)^2 - \alpha^4 \right].$$

The characteristic frequencies are then given by (exercise)

$$\omega_1 = \omega_0, \quad \omega_2 = \sqrt{\omega_0^2 + 2\alpha^2}.$$

The corresponding normalized normal modes are determined by

$$\vec{a}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{a}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We have

$$\Theta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t + \beta_1), \quad \Theta_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{\omega_0^2 + 2\alpha^2} t + \beta_2).$$

Evidently, the oscillation at frequency  $\omega_0$  corresponds to the two masses moving exactly in phase, so the coupling does not come into play, while the oscillation at frequency  $\sqrt{\omega_0^2 + 2\alpha^2}$  has the two masses moving exactly out of phase.

The general motion of the system is a superposition of these two normal modes. We can write the general motion of the system in vector form as

$$\vec{x}(t) = C_1 \vec{a}_1 \cos(\omega_1 t + \beta_1) + C_2 \vec{a}_2 \cos(\omega_2 t + \beta_2),$$

or, more explicitly,

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{2}} \left( C_1 \cos(\omega_1 t + \beta_1) + C_2 \cos(\omega_2 t + \beta_2) \right) \\ y(t) &= \frac{1}{\sqrt{2}} \left( C_1 \cos(\omega_1 t + \beta_1) - C_2 \cos(\omega_2 t + \beta_2) \right). \end{aligned}$$

Here  $(C_1, C_2, \beta_1, \beta_2)$  are real constants which are determined by initial conditions. Despite the fact that the motion is very regular, *i.e.*, it is the superposition of harmonic oscillations, the appearance of the motion can be quite complicated.

### Example: Double Pendulum

Let us return to the coplanar double pendulum. The Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + (m_1 + m_2)gl_1 \cos \theta_1 + m_2 gl_2 \cos \theta_2.$$

Recall that  $m_1$  has a fixed point of support, while  $m_2$  is supported at the location of  $m_1$ . Stable equilibrium occurs at  $\theta_1 = 0 = \theta_2$ . We expand in Taylor series about equilibrium to get the approximate Lagrangian:

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}(m_1 + m_2)gl_1\theta_1^2 - \frac{1}{2}m_2gl_2\theta_2^2.$$

Note that the coupling between degrees of freedom occurs through the kinetic terms. The characteristic equation is (exercise)

$$0 = \det \begin{pmatrix} (\omega^2 l_1^2 - gl_1)(m_1 + m_2) & \omega^2 m_2 l_1 l_2 \\ \omega^2 m_2 l_1 l_2 & (\omega^2 l_2^2 - gl_2)m_2 \end{pmatrix}.$$

The roots are (exercise)

$$\omega_{\pm}^2 = \frac{g}{2m_1 l_1 l_2} \left\{ (m_1 + m_2)(l_1 + l_2) \pm \sqrt{(m_1 + m_2) [(m_1 + m_2)(l_1 + l_2)^2 - 4m_1 l_1 l_2]} \right\}.$$

To carry on, let us consider a special case. Suppose the two pendula are identical:  $m_1 = m_2 = m$ ,  $l_1 = l_2 = l$ . Then the characteristic frequencies become

$$\omega_{\pm}^2 = \frac{g}{l}(2 \pm \sqrt{2}).$$

Note that all dependence upon the mass drops out. In this case the normal modes are determined by

$$\mathbf{a}_{\pm} = C \begin{pmatrix} 1 \\ \mp \sqrt{2} \end{pmatrix}.$$

The normal mode  $\Theta_+$  has both masses having velocities out of phase; the normal mode  $\Theta_-$  has the velocities in phase.

### Example: Linear triatomic molecule

As another application of our theory let us consider the motion near equilibrium of a linear triatomic molecule. By this we mean that we have two atoms of mass  $m$  located symmetrically on either side of an atom of mass  $M$ . Let all three atoms lie on a line. For simplicity we only consider longitudinal motion, *i.e.*, motion along the extent of the molecule. We assume that, near equilibrium, the restoring force on the atoms is  $-kx$ , where  $x$  is the displacement of an atom from equilibrium. In detail, let  $x_1$  and  $x_2$  denote the longitudinal displacements of each  $m$  from equilibrium, and let  $X$  denote the displacement of  $M$  from equilibrium. The potential energy is approximated by (exercise)

$$V = \frac{k}{2} [(X - x_1)^2 + (X - x_2)^2].$$

The kinetic energy is simply

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}M\dot{X}^2.$$

Ordering the coordinates on the configuration space as  $(x_1, x_2, X)$ , the matrices of interest are (exercise)

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix},$$

$$K = \begin{pmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{pmatrix}.$$

The characteristic equation is (exercise)

$$\omega^2(k - \omega^2 m)[k(M + 2m) - \omega^2 Mm] = 0.$$

The solutions are (exercise)

$$\omega_1 = 0 \text{ (!!!)}, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{k}{m} \left( \frac{M + 2m}{M} \right)}.$$

The appearance of a zero frequency mode is, at first, a little disconcerting, but simply reflects the possible motion in which all three masses move with a uniform translation (see below). The other two modes represent oscillations near equilibrium. To see all this, we need to compute the normal modes. We get (exercise)

$$a_1 = N_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad a_2 = N_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad a_3 = N_3 \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{M} \end{pmatrix},$$

where the  $N$ 's are normalization constants. The motion of mode 1 is rigid motion at constant velocity. To see this we simply note that

$$x_1 = x_2 = X = at + b$$

solves the equations of motion and with zero frequency (exercise). Note that a spatial translation of the system is a symmetry of the Lagrangian, so the total (center of mass) momentum is conserved. If desired, we can work in the rest frame of the center of mass; this reduction eliminates the center of mass degree of freedom and leaves two oscillatory degrees of freedom. The motion of mode 2 has the central atom at rest and the two endpoint atoms oscillating 180 degrees out of phase with the same amplitude. The third mode has the endpoint atoms in phase, with the same amplitude while the center atom



moves 180 degrees out of phase with them and with a different amplitude. These latter two modes keep the center of mass at rest.

As usual, the general motion of the molecule is a linear superposition of the normal modes with the coefficients of the superposition determined by initial conditions.

Finally, let us briefly and qualitatively consider the general, non-longitudinal motion of the molecule. Of course, if we allow for vibrations perpendicular to the line defined of the molecule we have more degrees of freedom (namely, 9) to consider. While the explicit computations are more lengthy, nothing conceptually new arises. There will be a number of zero frequency modes corresponding to rigid motions (translations and rotations) of the molecule. The remaining modes will be truly vibrational. Of the 9 degrees of freedom and the 9 corresponding modes, 5 will be of the zero-frequency type (exercise). Naively, there are 3 translations and 3 rotations, but the linear nature of the molecule, and its modeling via point masses, means that rotations about the molecular axis are not motions of the system. Thus there are only 5 zero frequency modes: 3 translations + 2 rotations. This leaves 4 vibrational modes. Two of these modes—the longitudinal modes—we have already studied. The other two modes are transverse to the axis of the molecule. If we think of this axis as the  $x$ -axis, then the transverse vibrations come from displacements in the  $y$  and  $z$  directions. Clearly there is nothing to physically distinguish  $y$  from  $z$ , so we expect that the frequencies for the two transverse normal modes will be degenerate. Indeed, there is nothing to pick out which orthogonal directions to the molecule should be  $y$  and  $z$ . The system exhibits a symmetry under rotations about the molecule axis. Thus the normal modes for transverse vibrations will be along any two perpendicular directions each of which is perpendicular to the molecule axis.

The rigid motions of the molecule, which give rise to the zero frequency modes, correspond to symmetries and conservation laws. Since the potential energy is changed only by a relative motion of the atoms, it clearly will exhibit a symmetry with respect to any transformation that leaves the relative position of the atoms unchanged, *i.e.*, rotations and translations of the molecule as a whole. The kinetic energy is, of course, invariant under such transformations (exercise). Thus the Lagrangian is invariant under the rigid rotations and translations of the molecule. The corresponding conservation laws are 5 in number (3 translations and 2 rotations—exercise); they are the 3 components of the center of mass momentum (no external forces) and 2 components of angular momentum (no external torques).

### **A Nice Mathematical Interpretation of Normal Modes: Simultaneous diagonalization of quadratic forms**

It is worthwhile giving an alternative description of the derivation of normal modes and

characteristic frequencies. This alternative description involves the notion of *simultaneous diagonalization of quadratic forms*. The idea is that the construction of the normal modes amounts to diagonalizing the matrices  $K$  and  $M$ , and the characteristic frequencies are coming from the diagonal entries of these matrices. As we shall see, strictly speaking,  $K$  and  $M$  are better thought of as quadratic forms—equivalently, symmetric tensors. One payoff of this slightly more sophisticated point of view is that it explains why we always get a basis of eigenvectors for  $M^{-1}K$  and hence why any motion of the dynamical system near equilibrium can be viewed as a superposition of the normal modes..

Recall that  $M$  can be viewed as assigning a number  $M(\vec{v})$  (twice the kinetic energy) to a vector  $\vec{v}$  (a velocity) via

$$M(\vec{v}) = M_{ij}v^i v^j.$$

The vectors (displacements, velocities) *etc.* are elements of a vector space. Denote the basis for the vector space as  $\vec{e}_i$ ,  $i = 1, 2, \dots, n$ . Consider a change of basis  $\vec{e}_i \rightarrow \vec{e}'_i$  in which the components of the vector  $\vec{v}$  change via

$$v^i \rightarrow v'^i = \Lambda^i_j v^j.$$

In a matrix notation:

$$v' = \Lambda v.$$

The kinetic energy — a physically observable quantity — cannot change under a change of basis; the components of  $M$  must therefore change as (exercise)

$$M'_{ij} = S_i^k S_j^l M_{kl},$$

or, in matrix notation,

$$M' = S^T M S,$$

where  $S = \Lambda^{-1}$ . This type of transformation of a matrix is called a *congruence transformation*.

Note that this is *not* the same transformation rule as that of a matrix representing a linear operator. A matrix  $L$  representing a linear operator transforms under a change of basis as a *similarity transformation*:  $L \rightarrow S^{-1}LS$ . Thus a quadratic form (equivalently, a symmetric rank 2 tensor) is not the same as a linear operator. If the change of basis is an orthogonal transformation, for which  $S^T = S^{-1}$ , then the distinction between transformation properties disappears.

Because the quadratic form  $M$  is positive definite, we can use it to define a scalar product:

$$(\vec{v}, \vec{w}) \equiv M(\vec{v}, \vec{w}) = M_{ij}v^i w^j.$$

*Exercise: Show that this does define a scalar product.*

As you know, we can always find an orthonormal basis for a vector space with scalar product. Thus we can always find a basis  $\vec{e}_i$  for which

$$M(\vec{e}_i, \vec{e}_j) = \delta_{ij}.$$

In this basis the kinetic energy is a sum of squares (exercise):

$$M(\vec{v}) = (v^1)^2 + (v^2)^2 + \dots + (v^n)^2.$$

Thus we have “diagonalized” the quadratic form defined by  $M$ .

Under the change to an orthonormal basis with respect to the scalar product defined by  $M$ , the matrix  $K$  representing the potential energy quadratic form will change by a congruence transformation to some other matrix, not necessarily diagonal. We now consider making a further change of basis characterized by a matrix  $S$  that keeps  $M$  unchanged from its diagonal form but which diagonalizes  $K$ . This change of basis that preserves  $M$  must be an orthogonal transformation since in the current basis

$$S^T M S = M \implies S^T S = I.$$

Under an orthogonal transformation the symmetric matrix  $K$  transforms via similarity transformation. But it is a standard result from linear algebra that every symmetric matrix can be diagonalized by a similarity transformation with an orthogonal matrix. Note, however, that this result does not allow us to turn  $K$  into the identity matrix, only a diagonal matrix. Thus we can always simultaneously diagonalize any two quadratic forms, provided one of them is positive definite so that it can be used to define a scalar product.

Here is a slightly different point of view of the last step in which  $K$  is diagonalized. In the basis in which  $M$  is the identity matrix but  $K$  is not yet diagonalized, our original computation of the characteristic frequencies and normal modes comes from solving

$$(\omega^2 I - K)\vec{a} = 0,$$

where  $K$  is the matrix computed arising from the congruence transformation that made  $M$  the identity. Evidently, in this basis we are just solving

$$K\vec{a} = \omega^2\vec{a},$$

*i.e.*, the eigenvalue problem for  $K$  in this basis. Finding the rotation which diagonalizes  $K$  is then essentially equivalent to solving this eigenvalue problem since the eigenvectors define an orthonormal basis in which  $K$  is diagonal.

A nice – if somewhat abstract – way to summarize this result is that  $M$  defines an inner product relative to which  $M^{-1}K$  is symmetric. Every symmetric operator on an inner product space admits a basis of eigenvectors. Finding these eigenvectors amounts to finding the normal modes.

This way of doing the analysis also has the virtue of making it clear the solutions one gets (vectors and frequencies) define a basis for the vector space. This guarantees that all solutions of the EL equations can be obtained from the normal modes by superposition.

When we work in the basis in which the two quadratic forms  $M$  and  $K$  are diagonal, we are using the directions defined by the normal modes to define new generalized coordinates. More precisely, if we make a change of generalized coordinates by making a change of basis that diagonalizes  $M$  and  $K$  as described above, then in this system of generalized coordinates (denoted  $Q^\alpha$ ,  $\alpha = 1, \dots, n$ ) the Lagrangian will have the form (exercise):

$$L = \frac{1}{2} \sum_{\alpha=1}^n \left( \dot{Q}^{\alpha 2} - \omega_\alpha^2 Q^{\alpha 2} \right).$$

The relation between the  $Q^\alpha$  and the normal modes  $\Theta_\alpha$  defined earlier is that the former are normalized so that the kinetic energy is just a sum of squares.

The relation between the normal modes  $Q^\alpha$  and the original coordinates  $x^i$  is as follows. Let the original vector space basis be denoted by  $\vec{b}_i$ ,  $i = 1, 2, \dots, n$ , and let the basis which diagonalizes  $M$  and  $K$  be denoted by  $\vec{e}_\alpha$  (these are the normalized eigenvectors). We have the change of basis matrices

$$\vec{b}_i = S_i^\alpha \vec{e}_\alpha, \quad \vec{e}_\alpha = S_\alpha^i \vec{b}_i.$$

We have

$$\vec{x} = x^i \vec{b}_i = Q^\alpha \vec{e}_\alpha,$$

so that

$$x^i = S_\alpha^i Q^\alpha.$$

## Forced Oscillations, revisited

As mentioned earlier, a typical scenario in which small oscillations are relevant is where one has a system in stable equilibrium which is subjected to an external force  $\vec{F}$  which moves the system from equilibrium. We allow this force to be time dependent, but we assume it does not depend upon the configuration variables of the system. This introduces an additional potential energy term  $V_1$  to the quadratically approximated Lagrangian given by

$$V_1 = -\vec{F}(t) \cdot \vec{x}.$$

Being linear in the displacement from equilibrium, this term is retained as is in the harmonic approximation.

Using the normal modes of vibration, the motion of the system can be reduced to a collection of uncoupled – but now forced – oscillators. Indeed, let us relate the original vector space basis, say  $\vec{b}_i$ , to that provided by the normal modes,  $\vec{e}_\alpha$  by the change of basis matrix  $S$ :

$$\vec{b}_i = S_i^\alpha \vec{e}_\alpha, \quad \vec{e}_\alpha = S_\alpha^i \vec{b}_i.$$

We then have

$$\vec{x} = x^i \vec{b}_i = Q^\alpha \vec{e}_\alpha,$$

where

$$Q^\alpha = S_i^\alpha x^i,$$

and

$$V_1 = -F_i(t)x^i = -F_\alpha(t)Q^\alpha,$$

where

$$F_\alpha(t) = S_\alpha^i F_i(t).$$

So the Lagrangian takes the form

$$L(Q, \dot{Q}, t) = \frac{1}{2} \sum_{\alpha=1}^n \left( \dot{Q}^{\alpha 2} - \omega_\alpha^2 Q^{\alpha 2} + F_\alpha(t) Q^\alpha \right).$$

We have  $n$  uncoupled, forced oscillators. We can therefore apply the technology we developed earlier for analytically displaying the forced oscillations to each normal mode.