Physics 6010, Fall 2016

Small Oscillations. Normal Modes.

Relevant Sections in Text: §6.1–6.3, 6.4

Small Oscillations: One degree of freedom.

We now leave the 2-body problem and consider another important class of systems which can be given a complete analytic treatment. I assume you already know that the motion of a system in the vicinity of a point of stable equilibrium is approximated by the superposition of harmonic oscillations. This approximation is very valuable and we shall spend some time studying it.

Let us begin by considering the motion of a one-dimensional system near a critical point of the potential energy. While only the simplest systems can be reduced to one degree of freedom, this case is still particularly important because – as we shall soon see – multi-dimensional systems (in the vicinity of stable equilibrium) can be reduced to multiple copies of the one dimensional case.

We assume the Lagrangian is of the form:

$$L = \frac{1}{2}a(q)\dot{q}^2 - V(q).$$

Note in particular that we assume for now that the system is autonomous, *i.e.,* energy is conserved. Let q_0 be a critical point of the potential, so that $V'(q_0) = 0$. Let us approximate the motion by assuming that $x := q - q_0$ is "small". We expand the Lagrangian in a Taylor series in x, keeping only the first non-trivial terms (exercise):

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \dots .$$

Here we have set $m = a(q_0)$ and $k = V''(q_0)$. Of course, the kinetic energy must be positive, so that $m > 0$. Note that we also have dropped the irrelevant constant $V(q_0)$. If desired we can adjust the zero point of potential energy so that this constant is zero.

The EL equations are (in the domain of validity of the approximation)

$$m\ddot{x} = -kx.$$

This equation is easily integrated:

$$x = Re(Ae^{i\omega t}), \quad \text{when } k \neq 0.$$

$$x = at + b, \quad \text{if } k = 0,$$

where $\omega = \pm\sqrt{\frac{k}{m}}$ and A is a complex constant encoding the two real integration constants, which can be fixed by initial conditions. If $k > 0$, then q_0 is a point of stable equilibrium, and we get harmonic motion. In particular, if x is small initially and the initial velocity is sufficiently small, then $x(t)$ remains small for all time (exercise), so that our approximation is self-consistent. On the other hand, if $k \leq 0$, then the motion of the particle need not maintain our approximation of small x; our approximation is not self-consistent and must be abandoned after a very short time. Of course, the case $k < 0$ corresponds to unstable equilibrium, for which a small perturbation leads to a rapid motion away from equilibrium. If $k = 0$, then the critical point q_0 is neither a maximum or a minimum but is a saddle point ("neutral equilibrium"); our approximation again becomes invalid, although the time scale for this is larger than the case of unstable equilibrium.

Thus, in a neighborhood of a point of stable equilibrium, it is consistent to make the *harmonic approximation* to the potential and kinetic energies. In the harmonic approximation the motion of the system is mathematically the same as that of a simple harmonic oscillator.

A very familiar example of all of this is the planar pendulum of mass m and length l for which $q = \theta$ is the deflection from a vertically downward position. We have

$$V(\theta) = mgl(1 - \cos\theta),$$

and

$$a(\theta) = \frac{1}{2}ml^2.$$

The Lagrangian in the harmonic approximation near equilibrium at $\theta = 0$ is (exercise)

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{mgl}{2}\theta^2,$$

so that the harmonic motion has "mass" ml^2 and angular frequency" $\omega = \sqrt{\frac{g}{l}}$. *(Exercise:* what happens to motion near $\theta = \pi$?)

Example

Consider a mass m which is constrained to move on a straight line. The mass is bound to a fixed point by harmonic force with potential energy $V = \frac{1}{2}K(r - R)^2$, where K is a constant, r is the distance of the particle to the fixed point. The distance from the point to the line is $l > R$. A mechanical model of this system is a mass sliding on a straight track; the mass being connected to a fixed point by a spring of equilibrium ("unstretched") length R. Our goal in this example is to find the stable equilibrium position(s) and compute the frequency of small oscillations about the equilibrium.

Evidently, $r^2 = x^2 + l^2$, where x is the position of the particle along the given line, with $x = 0$ the location at distance l from the center of force. Using x as the generalized coordinate, the Lagrangian for this system is (exercise)

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}K(\sqrt{x^2 + l^2} - R)^2.$$

You can check that $x = 0$ is the only point of stable equilibrium. (As an exercise you can check that the point $x = 0$ is a point of unstable equilibrium if we assume $R \geq l$.) You can expand the potential to second order to find that (exercise)

$$k = \frac{K(l - R)}{l},$$

so that

$$\omega = \sqrt{\frac{K(l - R)}{lm}}.$$

Note that $K(l - R)$ is the force needed to move a particle from $r = R$ to the point $r = l$ (exercise) in the presence of the potential V.

Example

We return to our example (from somewhat earlier in the course) of a plane pendulum of length l with horizontally moving point of support. The mass m_1 (the point of support) has position x, and the angular deflection of the pendulum (mass m_2) is denoted by ϕ. The kinetic energy is

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi),$$

and the potential energy is

$$V = -m_2 gl\cos\phi.$$

We have already seen that we can use conservation of the momentum conjugate to x to effect a Lagrangian reduction which eliminates the x degree of freedom. In detail, the conservation law we need is

$$P = (m_1 + m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi = \text{constant}.$$

This conservation law is just the conservation of the x component of the center of mass of the system (exercise).

Let us compute the frequency of small oscillations of the pendulum in the reference frame in which $P = 0$, so that (exercise)

$$(m_1 + m_2)x + m_2 l\sin\phi = \text{constant}.$$

In this reference frame, which is the rest frame of the x component of the center of mass, the reduced kinetic energy is (exercise)

$$T = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \frac{m_2}{m_1 + m_2} \cos^2 \phi\right),$$

and the reduced potential energy is (exercise)

$$V = -m_2 g l \cos\phi.$$

Clearly $\phi = 0$ is an equilibrium point (exercise). Expanding in powers of ϕ we get, in the harmonic approximation,

$$T \approx \frac{1}{2} \frac{m_1 m_2 l^2}{m_1 + m_2} \dot{\phi}^2,$$

$$V \approx \frac{1}{2} m_2 g l \phi^2.$$

It is now straightforward to see that the frequency of small oscillations is (exercise)

$$\omega = \sqrt{\frac{g(m_1 + m_2)}{m_1 l}}.$$

Of course, in a generic inertial reference frame the motion of the system in the harmonic approximation is small oscillations at the above frequency superimposed with a uniform translation of the center of mass along the x direction.

As a sanity check, let us consider the limit in which $\frac{m_2}{m_1} << 1$, *i.e.,* the mass at the point of support is becoming large. Physically, we expect that the point of support moves with a uniform translation along x. In the rest frame of the point of support (which is now the approximate center of mass) we have a traditional plane pendulum problem. In this limit we get (exercise)

$$\omega \approx \sqrt{\frac{g}{l}},$$

as expected.

Forced Oscillations

A typical scenario in which small oscillations is relevant is where one has a system in stable equilibrium which is subjected to an external force $\vec{F}$ which moves the system from equilibrium. We allow this force to be time dependent. This introduces an additional potential energy term V_1 to the quadratically approximated Lagrangian given by

$$V_1 = -\vec{F}(t) \cdot \vec{x}.$$

Let us explore this scenario in its simplest setting: a single degree of freedom. Later we will show how these results are used when there is more than one degree of freedom.

We thus consider the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + xF(t),$$

where $k > 0$. The equations of motion are that of a harmonic oscillator subjected to an external, time varying force $F(t)$ (exercise):

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}.$$

Of course, we must assume that F remains suitably small so that the solutions do not violate the approximation needed for their validity.

This inhomogeneous differential equation can be directly integrated as follows. Define

$$\xi(t) = \dot{x}(t) + i\omega x(t).$$

In terms of this complex variable the equation of motion takes the form (exercise)

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}.$$

You can easily see that, when $F = 0$, the solution is of the form $Ae^{i\omega t}$, where A is a constant. So try a solution of the form

$$\xi(t) = A(t)e^{i\omega t}.$$

Plugging into the ODE, we find that A satisfies

$$\dot{A} = e^{-i\omega t}\frac{F(t)}{m},$$

which has solution (exercise)

$$A(t) = \int dt\, \frac{1}{m}F(t)e^{-i\omega t} \; + \; \text{constant}.$$

Putting this all together, we see that the general solution of the forced oscillator equation is (exercise)

$$\begin{aligned} x(t) &= \frac{1}{\omega} Im(\xi(t)) \\ &= \frac{1}{\omega} Im\left[e^{i\omega t}(B + \int \frac{1}{m}F(t)e^{-i\omega t}\, dt)\right], \end{aligned}$$

where B is an arbitrary complex constant.

As an example, suppose that

$$F(t) = f\cos(\gamma t).$$

Then, provided $\gamma^2 \neq \omega^2$, the general solution is of the form (exercise)

$$x(t) = a\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t),$$

where a and α are real constants. We see that in this case the motion is a superposition of two oscillations at the two frequencies ω and γ inherent in the problem. The relative importance of the forced oscillation component depends, of course, on the the size of f, but also on the relative magnitudes of ω and γ.

When $\omega \to \gamma$ the forced oscillation amplitude diverges and our form of the solution given above becomes invalid; this situation is called *resonance.* To get the correct solution in this case we set $\gamma = \omega$ in our integral expression or the general solution. We then get a solution of the form (exercise)

$$x(t) = a\cos(\omega t + \alpha) + \frac{f}{2m\omega} t \sin\omega t,$$

where, again, a and α are constants. Note the linear growth in t, which eventually destroys the harmonic approximation.

We can get a useful picture of the behavior of the system near resonance by an approximation scheme. Let

$$\gamma = \omega + \epsilon,$$

where $\epsilon << \omega$. Write the general solution for $x(t)$ (off resonance) in the complex form (exercise):

$$\xi(t) \approx (A + Be^{i\epsilon t})e^{i\omega t}.$$

Over one period, $\frac{2\pi}{\omega}$, the amplitude $C = |A + Be^{i\epsilon t}|$ changes very little. Thus the motion is approximately that of free oscillation with a slowly varying amplitude. In particular, the amplitude is of the form (exercise)

$$C = \sqrt{a^2 + b^2 + 2ab\cos(\epsilon t + \phi)},$$

where $A = ae^{i\alpha}$, $B = be^{i\beta}$, and $\phi = \beta - \alpha$. Thus the amplitude varies (slowly) between the values $|a + b|$ and $|a - b|$. The oscillatory behavior is said to exhibit "beats".

Typically, a general force $F(t)$ can be Fourier analyzed into sinusoidal components. Likewise, we can Fourier analyze the solution $x(t)$. We can view the above example as illustrating the behavior of a typical Fourier component. The general motion of the system is then a superposition of motions such as given above (exercise).

Finally, let us note that since the Lagrangian for a system executing forced oscillations is explicitly time dependent (provided $\frac{dF}{dt} \neq 0$), there will be no conservation of energy for the oscillator. This should not surprise you, since the oscillator is clearly exchanging energy with its environment. We can compute the energy transferred during a time interval (t_1, t_2) by noting that the oscillator energy can be written as

$$E = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) = \frac{1}{2}m|\xi|^2,$$

and then using our explicit formula for $\xi(t)$,

$$\xi = e^{i\omega t}(B + \int_0^t \frac{1}{m}F(t)e^{-i\omega t}\, dt)$$

to compute the energy at time t. For example, let us suppose that the system is at equilibrium before $t = 0$, a force acts for a period of time after $t = 0$ after which the force is zero again. Then $B = 0$ and the change in the oscillator energy can be written as (exercise)

$$\Delta E = \frac{1}{2m}\left|\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\, dt\right|^2.$$

Thus, the energy transfer is controlled by the absolute value of the Fourier component of the force with frequency ω. If the time during which the force acts is small compared to $\frac{1}{\omega}$, then $e^{i\omega t}$ is approximately constant in the integral, and hence

$$\Delta E \approx \frac{1}{2m}\left|\int_{-\infty}^{\infty} F(t)\, dt\right|^2.$$

Here the change in energy is controlled solely by the impulse imparted by the force since the time scale is so short that no appreciable change in potential energy occurs while the force acts. In the limit where

$$F(t) = f\delta(t - t_0),$$

this approximation becomes exact (exercise).

Homework Problem

Consider an oscillator with mass m and natural frequency ω, initially at rest, which undergoes a constant force F_0 for a finite period of time T. Show that after the force ceases $(t > T)$ the system is oscillating harmonically. Determine the amplitude of this oscillation. Your answer should (only) depend upon F_0, m, ω, T .