Physics 6010, Fall 2016

The Kepler problem.

Relevant Sections in Text: §3.7 – 3.9, 3.12

Kepler Problem

The Kepler problem refers to the case where the potential energy is of the form

$$V(r) = \pm\frac{\alpha}{r}.$$

Here $\alpha > 0$, the plus sign corresponds to a repulsive force, such as between two charges of the same sign. The minus sign corresponds to an attractive force, such as the electric force between two oppositely charged particles, or the gravitational force between two point masses. In either case, it is straightforward to find the path the "particle"* takes, while the motion in time is more difficult to display explicitly.

We will consider the case of an attractive force. I will leave it as an exercise for you to translate, where appropriate, to the results for the case of a repulsive force. Note that the potential used in the Kepler problem is not sufficiently attractive to allow the "particle" to hit the origin when $l \neq 0$ (see the discussion of this point in the previous lecture). Thus, except in the exceptional case $l = 0$, the two bodies never collide. We will assume $l \neq 0$ in what follows.

Exercise: Describe the motion of the two bodies in the $l = 0$ case.

Let us first consider some general properties of the motion. The effective potential energy for the radial motion,

$$V_{\rm eff}(r) = -\frac{\alpha}{r} + \frac{l^2}{2mr^2},$$

tends to $+\infty$ as $r \to 0$, and tends to zero from negative values as $r \to \infty$, thus both bound (orbital, 2 turning points) and unbound (scattering, 1 turning point) motions are possible. There is a single, global minimum to the effective potential energy (exercise):

$$(V_{\rm eff})_{\rm min} = -\frac{m\alpha^2}{2l^2}.$$

This represents a stable equilibrium for the *radial* motion. If the energy and angular momentum are related via

$$E = (V_{\rm eff})_{\rm min}$$

* Keep in mind we are still considering the motion of two bodies; $\vec{r}$ is the relative position of the particles and r is the distance between them.

we get circular motion at a radius which you can work out from the previous equation to be

$$r = \frac{l^2}{m\alpha} = -\frac{\alpha}{2E}.$$

Let us next get an explicit formula for the orbital path. The orbital path is obtained by performing the integral relating ϕ and r; we get

$$\phi = \cos^{-1}\left\{\frac{\frac{l}{r} - \frac{m\alpha}{l}}{\sqrt{2mE + \frac{m^2\alpha^2}{l^2}}}\right\} + \text{constant}.$$

To keep things simple, let us adjust the origin of ϕ so that the integration constant vanishes. Define

$$p = \frac{l^2}{m\alpha}, \quad e = \sqrt{1 + \frac{2l^2E}{m\alpha^2}}.$$

Then we can write the path as (exercise)

$$\frac{p}{r} = 1 + e\cos\phi.$$

This is the equation of a *conic section* with the origin (where $r = 0$) being one of the foci. The parameter e is the *eccentricity* of the conic section. Evidently, our choice of origin for ϕ corresponds to the minimum value of r. If the motion is that of a body (*e.g.,* the Earth) and the sun, the point where $\phi = 0$ is the *perihelion*.

We see that the path of motion for the (attractive) Kepler problem is either circular ($e = 0$), elliptic ($e < 1$), parabolic ($e = 1$), hyperbolic ($e > 1$). From the expression for the eccentricity in terms of the energy, we see that circular motion does indeed correspond to the minimum of the potential energy (exercise):

$$E = -\frac{m\alpha^2}{2l^2},$$

as expected. Elliptic motion corresponds to

$$-\frac{m\alpha^2}{2l^2} < E < 0.$$

Parabolic motion corresponds to $E = 0$, and hyperbolic motion corresponds to $E > 0$.

If one body is very much more massive than the other, then it is easy to see that this massive body will remain (approximately) at the $r = 0$ focus of the conic section. For bound motion of a planet around the sun we thus recover Kepler's law of orbits.

For elliptical motion ($E < 0$) the semi-major axis has length (exercise)

$$a = \frac{p}{1 - e^2} = \frac{\alpha}{2|E|},$$

which only depends upon the energy of the particle. The semi-minor axis has length

$$b = \frac{p}{\sqrt{1-e^2}} = \frac{l}{\sqrt{2m|E|}},$$

which depends upon the energy and angular momentum.

The period of the motion for elliptical (or circular) orbits can be instructively computed as follows. Recall the equation defining the conserved magnitude of the angular momentum:

$$l = mr^2\dot{\phi}.$$

For infinitesimal displacements in time dt, it is straightforward to see that the quantity

$$df = \frac{1}{2}r^2 d\phi$$

is the area contained between the position vectors at time t and time $t + dt$. We call

$$\dot{f} = \frac{1}{2}r^2\dot{\phi}$$

the *areal velocity* since it represents the rate at which area is swept out by the position vector in time. Conservation of angular momentum tells us that the areal velocity is constant. Thus we recover Kepler's law of areas: equal areas are swept out in equal times. Note that this result follows from conservation of angular momentum and so is not specific to the Kepler problem; it holds for any 2-body central force problem. Now, we have

$$l = 2m\dot{f}.$$

In one period the position vector sweeps out the area πab of the ellipse. Thus the period T is given by (exercise)

$$T = \frac{2m\pi ab}{l} = \pi\alpha\sqrt{\frac{m}{2|E|^3}}.$$

Note that the period only depends upon the energy. Since the semi-major axis is proportional to $\frac{1}{|E|}$, we see Kepler's law of periods being verified (exercise).

When $E \geq 0$ the motion is unbound. When $E > 0$, we have $e > 0$ and the motion is hyperbolic with perihelion

$$r_{\text{min}} = \frac{\alpha(e-1)}{2E},$$

which depends on both energy and angular momentum (exercise). When $E = 0$, we have $e = 1$ and

$$r_{\text{min}} = \frac{l^2}{2m\alpha}.$$

Note that parabolic motion has the "particle" having zero velocity asymptotically (exercise).

We have been focusing primarily on the path of the "particle". Its motion in time is governed by Kepler's law of areas, as we have discussed. Unfortunately, a completely explicit analytical description of the motion is hard to come by. In particular, the integral giving $t(r)$ is tractable enough, but the problem of explicitly inverting this relationship to get $r(t)$ (which is also needed to get $\phi(t)$) is somewhat involved. See your text for details.

A hidden conservation law

We have mentioned that the Kepler (and isotropic oscillator) problems are special among 2-body problems in that the bound motions are *closed* orbits. This special feature arises because the potentials $V \propto \frac{1}{r}$ (and $V \propto r^2$) allow for an extra Lagrangian symmetry beyond the obvious spacetime translations and rotations. Noether then tells us that there is a conserved quantity, and it turns out this conservation law keeps the orbits closed.

In the Kepler problem, the extra conservation law is vectorial in nature and is usually called the "Laplace-Runge-Lenz vector". It is given by

$$\vec{A} = m\dot{\vec{r}} \times \vec{l} - \alpha m \frac{\vec{r}}{r}.$$

As a nice exercise you can check that this quantity is conserved by the Kepler motion. The symmetry responsible for this conservation law is far from obvious. Using some of the more advanced aspects of Noether theory the infinitesimal form of the symmetry can be obtained from the above form of the conservation law. I will spare you the details, but it's amusing to write symmetry down. Let $\vec{\lambda}$ be any constant vector (in time). Then the conservation of $\vec{\lambda} \cdot \vec{A}$ is implied by the following Lagrangian symmetry:

$$\delta\vec{r} = 2(\vec{\lambda} \cdot \vec{r})\dot{\vec{r}} - (\vec{\lambda} \cdot \dot{\vec{r}})\vec{r} - (\vec{r} \cdot \dot{\vec{r}})\lambda.$$

I won't ask you to check that this *is* a Lagrangian symmetry. Try it only if you feel strong. There is an analogous symmetry and conservation law for the isotropic oscillator.

The vector $\vec{A}$ represents 3 conservation laws. But there is really only one *new* conservation law here. To see this, we just have to do a little vector algebra. Firstly, you can check that

$$\vec{l} \cdot \vec{A} = 0.$$

This means that $\vec{A}$ is always in the plane orthogonal to $\vec{l}$, *i.e.,* the orbital plane of the two bodies. Thus one of the three components of $\vec{A}$ is actually determined by $\vec{l}$. Secondly, you can easily check that

$$A^2 = m^2\alpha^2 + 2mEl^2,$$

so that the magnitude of $\vec{A}$ is in fact determined by the energy and angular momentum. Thus there is a single conservation law in $\vec{A}$ independent of $\vec{l}$ and E – geometrically this conservation law is the *direction* of $\vec{A}$ in the plane orthogonal to $\vec{l}$.

The conservation of $\vec{A}$, involving as it does conservation of $\vec{l}$ and E, can in fact be used to obtain the equation for orbits. Moreover, we can see that the new conservation law (the direction of $\vec{A}$ in the plane orthogonal to $\vec{l}$) is responsible for the closure of bound orbits.

Consider the quantity $\vec{A}\cdot\vec{r}$:

$$\vec{A}\cdot\vec{r} = Ar\cos\theta.$$

Here θ is the angle between the position of the "particle" and the (unchanging in time) vector $\vec{A}$. Explicit computation yields (exercise)

$$\vec{A}\cdot\vec{r} = l^2 - m\alpha r.$$

From the preceding two equations we see that (exercise)

$$\frac{1}{r} = \frac{m\alpha}{l^2}\left(1 + \frac{A}{m\alpha}\cos\theta\right).$$

But this is just the orbit equation! Thus the conservation of $\vec{A}$ deterimines the orbital motion. In particular, we see that

$$A = m\alpha e.$$

Apparently, $\vec{A}$ points from the origin (a focus of the ellipse) along the major axis toward perihelion (exercise). Thus the fact that the direction of $\vec{A}$ is conserved corresponds to the fact that the perihelion cannot change from orbit to orbit, which implies that the orbit is closed (exercise).

Finally, it is worth noting that the existence of this "hidden conservation law" in the Kepler problem is responsible for the extra "accidental" degeneracy in the energy spectrum of the hydrogen atom. Recall that for a central force problem in quantum mechanics the energy $E_{n,l}$ is defined by a "principal quantum number", n, and the total angular momentum quantum number l. Because the Hamiltonian is rotationally invariant, one will have $2l+1$ different states with the same energy $E_{n,l}$. For the hydrogen atom, however, the energy E_n is in fact independent of l, which must lie in the range $l = 0, 1, \ldots, n-1$. Thus the hydrogen atom has an additional "accidental" degeneracy so that the actual degeneracy of the E_n is $n^2 \geq 2l+1$. This additional degeneracy peculiar to the hydrogen atom (and to the isotropic oscillator) reflects the existence of an additional conservation law *i.e.,* of an extra operator that commutes with the Hamiltonian. This additional operator is contained in the Laplace-Runge-Lenz vector. Any departure from a $\frac{1}{r}$ potential will destroy this conservation law and remove the extra degeneracy.

A few comments on the 3-body problem

We have seen that the conservation laws arising in the 2-body central force problem allow us to completely find the motion of the system "up to quadrature". What happens when there are more bodies? Here we make a few comments on the nature of the 3-body central force problem.

Simple counting of degrees of freedom and conservation laws suggests this problem is going to be much more complex than the 2-body problem. We still expect conservation laws of energy, momentum and total angular momentum – leading to a reduction in 6 degrees of freedom. But now, of course, we have 9 degrees of freedom, so conservation laws – while useful – can't be used to completely determine the motion of the system.

You can easily write down the Lagrangian and EL equations for 3 masses interacting via central potentials. These equations are put into a particularly nice form in your text, §3.12. Here I just want to comment on a simple model problem: the *restricted 3-body problem*. The idea is to consider two masses much larger than a third mass, which can then be viewed as a "test particle", *i.e.,* moving in the gravitational field of the more massive bodies, but having negligible effect on their motion. The motion of the two large masses is then completely understood. For simplicity, let us assume that motion is circular motion in the x-y plane about the center of mass with frequency ω. It is a good idea then to work in a rotating reference frame in which the two more massive bodies are at rest. In this reference frame one can write down the Lagrangian for the test particle; it includes coriolis and centrifugal terms (see your text).

There are some interesting equilibrium points ("Lagrange points") for the restricted 3-body problem. They can be analyzed as follows. Recall that three points always determine a plane. So, at any instant of time, there is a plane containing the 3 masses. Limiting attention once again on circular motion of the two more massive bodies, let this be the $x-y$ plane in the rotating coordinates mentioned above. One can then compute the potential energy for the test body as a function of x and y, and find equilibrium points, study their stability, *etc.* See your text for details.