Physics 6010, Fall 2016

First integrals. Reduction. The 2-body problem.

Relevant Sections in Text: §3.1 – 3.5

Explicitly Soluble Systems

We now turn our attention to some explicitly soluble dynamical systems which are very important in mechanics. The systems we will study include one dimensional systems, two body central force systems, and linear (oscillatory) systems. In each of these cases the explicit solubility of the equations of motion arises from the existence of suitable symmetries/conservation laws. The method by which we solve for the motion of these systems involves an important mechanics technique we shall call *reduction*.

First integrals and motion in one dimension

We turn our attention to a class of dynamical systems which can always be understood in great detail: the autonomous system* with one degree of freedom. For autonomous systems with one degree of freedom the motion is determined by conservation of energy. We shall explore this for Newtonian systems, *i.e.,* for systems described by Lagrangians of the form $L = T - V$.

Often times the study of motion of a dynamical system can be reduced to that of a one-dimensional system. For example, the motion of a plane pendulum reduces to dynamics on a circle. Another example that we shall look at in detail is the 2-body central force problem. Finding the motion of this system amounts to solving for the motion of the relative separation of the two bodies, which is ultimately a one-dimensional problem. No matter how the one-dimensional system arises, in very general circumstances we can completely solve the equations of motion.

Although we could proceed more generally, we will restrict our attention to the very common situation where the Lagrangian for a one-dimensional system is of the form

$$L = \frac{1}{2}a(q,t)\dot{q}^2 - V(q,t),$$

where q is the generalized coordinate for the system and a, V are some given functions. For example, a plane pendulum of length l and mass m has (exercise)

$$a(q,t) = ml^2,$$

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* Recall that a dynamical system is "autonomous" if its equations of motion do not have explicit time dependence.

and

$$V(q,t) = -mgl\cos q.$$

Here q is the angular position of the pendulum from the vertical.

As another example, consider a plane pendulum whose length oscillates according to $l = l_0 \cos\gamma t$. In this case (exercise)

$$a(q,t) = ml_0^2\cos^2(\gamma t),$$

and

$$V(q,t) = -mgl_0\cos(\gamma t)\cos(q).$$

Here we see how the kinetic and potential energies can get an explicit time dependence.

Yet another example arises when we consider a particle moving on a curve $y = 0$, $z = f(x)$ under the influence of gravity (a roller coaster!). This system has (exercise)

$$a(q,t) = m(1 + f'^2)$$

and

$$V(q,t) = mgf(q),$$

where we have chosen the position on the x-axis to be q. Here we see how the kinetic energy can become a function of the generalized coordinate.

Our main result is the following. For Lagrangians of the form

$$L = \frac{1}{2}a(q)\dot{q}^2 - V(q),$$

i.e., with no explicit time dependence, the equations of motion can be solved "up to quadrature". This means that the solution can be found using only integration and algebra. The proof uses conservation of energy. Since $\frac{\partial L}{\partial t} = 0$, we have that

$$E = \frac{1}{2}a(q)\dot{q}^2 + V(q)$$

is constant (time independent) when evaluated on a solution to the equations of motion (EL equations). Thus we have (exercise)

$$\frac{1}{2}a(q(t))\left(\frac{dq}{dt}\right)^2 + V(q(t)) = \text{constant} \equiv E.$$

For a given value of E, we thus get a first-order (albeit non-linear) ODE for $q(t)$. Since the EL equations are second-order (exercise) we have, in effect, used conservation of energy to integrate the equations of motion once. For this reason one often calls E a *first integral* of the equations of motion. Another way to see the origin of this terminology comes from writing the conservation of energy as the identity built from the EL equations:

$$\frac{d}{dt}\left[\frac{1}{2}a(q)\dot{q}^2 + V(q)\right] = \dot{q}\left(V' - \frac{1}{2}a'\dot{q}^2 + \frac{d}{dt}(a\dot{q})\right),$$

where

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -\left\{V' - \frac{1}{2}a'\dot{q}^2 + \frac{d}{dt}(a\dot{q})\right\}.$$

You can see from this identity that if we multiply the EL equations of motion by $-\dot{q}$ then we can write the result as the derivative of E.

Conservation of energy replaces the EL equation with a first-order ODE, which can be solved as follows. Fix a numerical value for E; this is equivalent to partially fixing the initial conditions.* We then rearrange the energy conservation equation given above into the form

$$dt = \pm\sqrt{\frac{a(q)}{2(E - V(q))}}\,dq,$$

so that, upon integrating both sides,

$$t - t_0 = \pm\int_{q_0}^{q}\sqrt{\frac{a(x)}{2(E - V(x))}}\,dx.$$

Here q_0 is the other integration constant (besides E) which arises when solving a second order ODE. t_0 is free parameter corresponding to a choice of initial time; when $t = t_0$ we have $q = q_0$. Note also that we have a sign choice to make when taking the square root. This choice specifies whether the velocity is initially toward increasing or decreasing q. with an increase in time. Both behaviors are possible in general; the boundary between these two types of motion is a time where the velocity vanishes. By specifying the $\pm$ sign, the value of q_0, and the value of E we have completely specified the solution. Thus these choices are equivalent to specifying the initial values of q and $\dot{q}$ (exercise).

By performing the indicated integral we get a formula $t = t(q)$, which also depends upon the initial position, initial time, and energy. To get the solution of the equations of motion, $q = q(t)$, we need to invert this formula. This can be done provided $\frac{dt}{dq}$ is finite and non-vanishing (exercise). This is guaranteed by the equations of motion away from points where $E = V$, *i.e.,* where the kinetic energy vanishes. As we shall see, these are isolated points. Away from such points we can solve (in principle) for $q = q(t)$, which also depends upon the initial position, initial time, and energy. By continuity we get

* The EL equations are a single second order ODE. Solutions are uniquely determined by 2 numbers, corresponding, *e.g.,* to the initial values of q and $\dot{q}$. By fixing a value for the energy we are putting one condition on the initial q and $\dot{q}$ values.

the solution everywhere. We thus get the solution to the equations of motion depending upon two constants of integration (q_0 and E). We also have the $\pm$ sign in the solution. Specifying q_0 and E and a choice of sign is equivalent to specifying the initial position and velocity (exercise). Thus we obtain the general solution to the equations of motion.

Motion in One Dimension: Examples

For sufficiently simple forms of $a(q)$ and $V(q)$ a closed form expression for the motion of the system can be found. Let us look at some elementary examples.

Free Particle

Here $V(q) = 0$ so that $L = \frac{1}{2}mv^2$, hence

$$\begin{aligned} t - t_0 &= \pm \int_{q_0}^{q} \sqrt{\frac{m}{2E}}\, dx \\ &= \pm\sqrt{\frac{m}{2E}}(q - q_0). \end{aligned}$$

We thus recover the familiar free particle motion (exercise)

$$q = q_0 + v_0(t - t_0),$$

where

$$v_0 = \pm\sqrt{\frac{2E}{m}}$$

is the initial velocity.

Constant Force

We choose $V(q) = -Fq$ where F is the constant force on the particle so that $L = \frac{1}{2}m\dot{q}^2 + Fq$. We find (exercise)

$$\begin{aligned} t - t_0 &= \pm \int_{q_0}^{q} \sqrt{\frac{m}{2(E + Fx)}}\, dx \\ &= \pm\frac{1}{F}\sqrt{2m}(\sqrt{E + Fq} - \sqrt{E + Fq_0}). \end{aligned}$$

You can easily check that this leads to the usual quadratic dependence of q on time (exercise).

Harmonic Oscillator

A simple harmonic oscillator is treated via

$$\begin{aligned} t - t_0 &= \pm \int_{q_0}^{q} \sqrt{\frac{m}{2(E - \frac{1}{2}kx^2)}}\, dx \\ &= \pm\sqrt{\frac{m}{k}} \left[\sin^{-1}\left(\sqrt{\frac{k}{2E}} q\right) - \sin^{-1}\left(\sqrt{\frac{k}{2E}} q_0\right) \right]. \end{aligned}$$

We can thus write the solution in the familiar form (exercise)

$$q(t) = A\sin(\omega t + \alpha),$$

where

$$\omega = \pm\sqrt{\frac{k}{m}}$$

$$A = \sqrt{\frac{2E}{k}}$$

and

$$\alpha = \sin^{-1}\left(\frac{q_0}{A}\right).$$

More complicated potentials may or may not be amenable to analytic computations, that is, explicity evaluation of the integral. For example, the simple plane pendulum has $V(q) = mgl\cos q$ and this leads to an elliptic integral (exercise), which is not as tractable analytically as the previous examples. Still, one can extract whatever information is needed from the elliptic integral, if only numerically. Of course, in the small oscillation approximation this integral reduces to our harmonic oscillator example.

Motion near equilibrium

Let us consider the motion near a stable equilibrium. Stable equilibrium occurs at points q_0 where

$$V'(q_0) = 0, \quad V''(q_0) > 0.$$

Of course, if we start the system off so that $E = V(q_0)$, then the velocity is zero and must remain zero for all time (exercise). If the system has $E - V(q_0) = \epsilon << 1$, $\epsilon > 0$, then the particle never goes very far from the equilibrium position. In this case we can approximate the potential as

$$V = V(q_0) + \frac{1}{2}V''(q_0)(q - q_0)^2$$

and we can approximate

$$a(q) \approx a(q_0).$$

With this approximation, we are back to our harmonic oscillator example with the substitutions

$$m \to a(q_0), \quad k \to V''(q_0), \quad E \to \epsilon.$$

In particular, the motion is not just periodic but harmonic (frequency independent of amplitude) with frequency

$$\omega = \sqrt{\frac{V''(q_0)}{a(q_0)}}.$$

Example: A Falling Rod

Let us work a non-trivial example in a little detail. Consider a thin rigid rod of mass m and length L standing upright in a uniform gravitational field. This is a state of unstable equilibrium, the slightest perturbation of its position or horizontal velocity will cause the rod to fall. Let us compute the time it takes to fall. We make the simplifying assumption that the base of the rod does not move relative to the ground. Thus the system has one degree of freedom corresponding to rotation of the rod about an axis perpendicular to and through the bottom end of the rod. We choose as our generalized coordinate the angle θ between the rod and the ground. The kinetic and potential energies are (exercise)

$$T = \frac{1}{6} m L^2 \dot{\theta}^2, \quad V = \frac{mgL}{2} \sin\theta.$$

The conserved total energy is $T + V$ so that, for initial conditions characterized by θ_0 and E, we have

$$t - t_0 = \pm \int_{\theta_0}^{\theta} dx \sqrt{\frac{\frac{1}{3} m L^2}{2(E - \frac{mgL}{2}\sin x)}} = \pm \sqrt{\frac{L}{3g}} \int_{\theta_0}^{\theta} dx \sqrt{\frac{1}{\alpha - \sin x}},$$

where

$$\alpha = \frac{2E}{mgL}.$$

This results in an expression involving elliptic integrals of the second kind, which is not surprising since our model of the falling rod is just a pendulum. Let us consider the time it takes for the rod to fall to the ground. This means θ should decrease in time; we need the minus sign in the above formula. Just to get some numbers, let us choose $L = 2m$, $\alpha = 1$, $\theta = 0$, and $\theta_0 = \frac{\pi}{2} - 0.1$ radian. This corresponds to a very small initial displacement from the vertical and a very small initial velocity. We then get (exercise) $t - t_0 = 1.04s$.

Qualitative Description of the Motion

We now indicate some general, semi-qualitative techniques for analyzing the motion of autonomous systems with one degree of freedom We are still assuming that the Lagrangian has no explicit time dependence. Conservation of energy determines the motion. Let us assume that $a(q) > 0$. Because the kinetic energy is positive and the solutions to the equations of motion must be real, it follows that the value of q must be such that total energy cannot be less than the potential energy:

$$V(q) \leq E.$$

For a given value of the energy E, the isolated points where $E = V$ are called *turning points* since the velocity must vanish there. Generically, the velocity changes direction (as time increases) at a turning point, keeping the "particle" in the domain where $E \geq V(q)$. If the energy is such that the allowed region of motion is surrounded by a pair of turning points, then the motion is *bound* and periodic (exercise). If the energy is such that a given region only has one or no turning point, then the "particle" will move to infinity, perhaps after encountering a turning point. We say that the motion is *unbound*. Exceptional cases occur when the energy is such that the turning point is a maximum or minimum of the potential. In this case the motion is that of a (un)stable equilibrium or is such asymptotically.

For bound motion it is easy to see that the motion must be periodic (exercise).* The period τ of the motion can be computed from our integral expression for the solution to the equations of motion. For a given value E of the energy we have (exercise)

$$\tau(E) = \int_{q_1}^{q_2} \sqrt{\frac{2a(x)}{E - V(x)}}\, dx,$$

where the values q_1 and q_2 are the turning points of the bound motion, *i.e.,* for a given E, $V(q_1) = V(q_2) = E$. Of course, the existence of two roots is guaranteed by our assumption that the motion is bounded.†

As a simple example of this latter formula, let us return to the simple harmonic oscillator. The turning points are (exercise)

$$q_1 = -\sqrt{\frac{2E}{k}}, \quad q_2 = \sqrt{\frac{2E}{k}}.$$

* Note that "periodic" does not mean "harmonic". In general the period of the motion depends upon the amplitude, *i.e.,* the turning points.

† As a very nice exercise, you should be able to show that – unless the turning point is at a critical point of the potential – the period is always finite. This is not immediately obvious since the integrand blows up as you approach a turning point. What happens physically when the integral *does* diverge?

We thus get (exercise)

$$\begin{aligned}\tau(E) &= \int_{-\sqrt{\frac{2E}{k}}}^{\sqrt{\frac{2E}{k}}} \sqrt{\frac{2m}{E-\frac{1}{2}kx^2}}\, dx,\\ &= 2\sqrt{\frac{m}{k}}[\sin^{-1}(1)-\sin^{-1}(-1)]\\ &= 2\pi\sqrt{\frac{m}{k}}.\end{aligned}$$

To summarize, when the Lagrangian for a one-dimensional system is of the form

$$L = \frac{1}{2}a(q)\dot{q}^2 - V(q),$$

the motion can be completely determined and involves the familiar cases of periodic bound motion, unbound motion, as well as stable and unstable equilibria. This very regular type of motion arises because conservation of energy is "running the show". If we drop conservation of energy, *i.e.,* we let t appear in the Lagrangian, then the motion of the system can be considerably more complicated. Indeed, already in the relatively simple setting of a system with one degree of freedom *and* a time dependent Lagrangian, it is possible to find chaotic motion.

Two Body Central Force Problem

We have already mentioned the two body central force problem several times. Here we shall begin a systematic study of this important dynamical system. As we shall see, the conservation laws admitted by this system allow for a complete determination of the motion (up to quadrature). Many of the topics we have been discussing in previous lectures come into play here. While this problem is *very* instructive and physically quite important, it is worth keeping in mind that the complete solvability of this system makes it an exceptional type of dynamical system. We cannot solve for the motion of a generic system as we do for the two body problem.

The two body problem involves a pair of particles with masses m_1 and m_2 described by a Lagrangian of the form:

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - V(|\vec{r}_1 - \vec{r}_2|).$$

This Lagrangian is invariant under spatial translations, time translations, rotations, and boosts.* Thus, as discussed earlier, we will have conservation of total energy, total momentum and total angular momentum for this system.

* "Boosts" are transformations corresponding to changing to a reference frame moving with constant relative velocity. The totality of these symmetry transformations (spacetime translations, rotations, boosts) in non-relativistic Newtonian mechanics forms a group known as the *Galileo group.*

The EL equations are equivalent to (exercise)

$$m_1\ddot{\vec r}_1 = V'(|\vec r_1 - \vec r_2|)\,\frac{\vec r_2 - \vec r_1}{|\vec r_1 - \vec r_2|},$$
$$m_2\ddot{\vec r}_2 = V'(|\vec r_1 - \vec r_2|)\,\frac{\vec r_1 - \vec r_2}{|\vec r_1 - \vec r_2|}.$$

Here the prime on the potential energy function denotes a derivative with respect to its argument. The equations of motion consist of 6 coupled ODEs corresponding to the 6 degrees of freedom. In general these equations are non-linear.*

From these equations we see that the force exerted on each particle (i) is equal in magnitude and opposite in direction to that on the other particle (*i.e.,* this force obeys Newton's third law); (ii) is directed along a line joining the (instantaneous) particle positions, and (iii) has a magnitude depending only on the distance between the particles.

Of course, the standard example of such a two body problem arises in the motion of gravitating bodies ("Kepler problem"), or in the motion of a pair of electric charges ignoring magnetic and radiative effects. In each case the potential energy is of the form

$$V \propto \frac{1}{|\vec r_1 - \vec r_2|}.$$

The constant of proportionality depends upon the masses, charges, gravitational constant, *etc.*

Reduction to relative motion

Our first step toward solving the two body problem is to reduce the problem to that of the relative motion of the two bodies. The physical idea is simple. Given the homogeneity of space, the absolute position of either of the particles has no relevant meaning. Only the relative position should matter. This idea manifests itself as follows. Because the system is invariant under spatial translations, total momentum will be conserved. This means the motion of the center of mass will be at constant velocity in any inertial reference frame (IRF) (exercise). So, this conservation law tells us that 3 degrees of freedom move as a free particle. The remaining 3 degrees of freedom describe the relative motion of the particles; it is here where all the interesting dynamics lie. This effective reduction in degrees of freedom from 6 to 3 as arises from the conservation of total momentum (which is the center of mass momentum) of the system.

Let us now make the above comments mathematically precise. Make a change of generalized variables (*i.e.,* a point transformation):

$$\vec R = \frac{m_1\vec r_1 + m_2\vec r_2}{m_1 + m_2},$$

* There is one exceptional case where the equations are linear. Can you see what this case is?

$$\vec{r} = \vec{r}_1 - \vec{r}_2.$$

We call $\vec{R}$ the *center of mass position* and we call $\vec{r}$ the *relative position.* This is easily seen to be an invertible transformation (exercise). We can express the Lagrangian as (exercise)

$$L = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}m\dot{\vec{r}}^2 - V(|\vec{r}|),$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

is the *reduced mass* of the system. We see that, in these coordinates, the Lagrangian is of the form

$$L(\vec{r}, \vec{R}, \dot{\vec{r}}, \dot{\vec{R}}) = L_{cm}(\dot{\vec{R}}) + L_{relative}(\vec{r}, \dot{\vec{r}}),$$

indicating that the center of mass motion $\vec{R}(t)$ and relative motion $\vec{r}(t)$ are completely decoupled (exercise). We can solve for the motion of $\vec{R}$ and $\vec{r}$ separately. The motion of $\vec{R}$ is obvious:

$$\vec{R}(t) = \vec{R}_0 + \vec{V}t.$$

By a suitable choice of inertial reference frame we can even set $\vec{R}(t) = 0$ without affecting $\vec{r}(t)$ (exercise). Put differently, up to a boost — which is a symmetry for this system — all solutions to the equations of motion can be obtained by setting $\vec{R}(t) = 0$ and solving for the relative motion. Once we have determined $\vec{r}(t)$, we can easily reconstruct $\vec{r}_1(t)$ and $\vec{r}_2(t)$ from $\vec{r}(t)$ and $\vec{R}(t)$ (exercise).

In any case, the only non-trivial problem to solve is the relative motion problem. We henceforth focus on the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(|\vec{r}|),$$

which describes the relative motion. Note that, mathematically speaking, this dynamical system is indistinguishable from a single particle moving in three dimensions subject to a central force field directed at fixed point in space. We have thus used the conservation of total momentum to reduce the analysis of the motion of two bodies interacting by a central force to the analysis of a system mathematically equivalent to that of a single particle moving in a central force field. All our subsequent analysis will apply to this physical system. Keep in mind that, in applications to the 2-body problem, the "particle" with mass m and position $\vec{r}$ is fictitious: its position is really the relative position of the two bodies, and the "particle" mass is really the reduced mass of the two body system.

Exercise: Show that when $m_2 >> m_1$ we can work in a reference frame in which m_2 is (approximately) at rest and we can view the reduced Lagrangian as describing the motion of m_1 in the fixed force field of m_2.

A small digression: Lagrangian Reduction

By a clever change of variables in the 2 body problem, $(\vec{r}_1, \vec{r}_2) \to (\vec{R}, \vec{r})$, we are able to obtain a Lagrangian in which three coordinates, namely (R_x, R_y, R_z), are cyclic. Thus the center of mass degrees of freedom "move" according to free particle EL equations and can be eliminated from the problem leaving us with a reduced problem involving 3 fewer degrees of freedom. You can see now why cyclic coordinates are also sometimes called "ignorable".

The reduction of the 2 body problem to a problem involving only 3 degrees of freedom by using conservation of total momentum is an instance of what is usually called *Lagrangian reduction*. This term describes the effective reduction in degrees of freedom afforded by a conservation law. A trivial example of Lagrangian reduction occurs already for motion in one-dimension q described by a time independent Lagrangian. The conservation of energy allows us to completely solve for $q(t)$ (up to quadrature)—thus leaving no more degrees of freedom to worry with. As we shall see, the complete integration of the EL equations for the 2-body problem is nothing but a sequence of Lagrangian reductions. There is a very general and powerful theory of reduction in mechanics (and field theory), but it will take us too far afield to develop it systematically. The best version of it takes place in the Hamiltonian formalism (to be discussed later), where the technique is called "symplectic reduction". Still, even though we will refrain from developing the general theory of Lagrangian reduction, because this is such an important tool in mechanics it is worth digressing for a moment to look at a couple of other examples of this phenomenon.

Example: Spherical Pendulum

The Lagrangian is

$$L = \frac{1}{2} m R^2 (\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) + mgR\cos\theta.$$

The coordinate ϕ is ignorable – it does not appear in L – and hence (exercise)

$$mR^2 \sin^2\theta \dot{\phi} = l = \text{constant}.$$

If we fix a value for the constant l (part of the specification of initial conditions) this equation can be used to eliminate the ϕ degree of freedom in terms of an integral involving $\theta(t)$:

$$\phi(t) = \int_{t_0}^{t} ds \frac{l}{mR^2 \sin^2\theta(s)},$$

Thus we have reduced the problem to understanding time evolution of a single degree of freedom, θ. Once we find $\theta(t)$ we can easily obtain $\phi(t)$ by an integral, and the problem is solved.

The motion in θ, *i.e.,* $\theta(t)$, is determined by the equation of motion for θ, with $\phi(t)$ eliminated using the conservation law above (exercise). This can be done because $\phi(t)$ does not appear in the equation, only $\dot\phi(t)$. This is because only $\dot\phi$ appears in the Lagrangian. Thus the cyclic coordinate ϕ implies a conservation law *and* a special feature of the θ equation of motion, namely, that it only involves $\dot\phi$. These two effects of cyclic ϕ lead to the reduction we are discussing. The equation of motion for θ is equivalent to conservation of energy, so alternatively, we can eliminate $\dot\phi$ in the energy:

$$E = \frac{1}{2}mR^2\dot\theta^2 + \frac{1}{2}\frac{l^2}{mR^2}\sin^2\theta - mgR\cos\theta,$$

and use the conservation of energy to find $\theta(t)$ as described earlier.

Thus we use the conservation of angular momentum to reduce the two degrees of freedom of the pendulum to an effective one-dimensional system which can be solved via conservation of energy.

Example: Plane pendulum with moving point of support.

The Lagrangian is

$$L = \frac{1}{2}(m_1+m_2)\dot x^2 + \frac{1}{2}m_2\left(2l\dot x\dot\phi\cos\phi + l^2\dot\phi^2\right) - m_2gl\cos\phi.$$

The coordinate x is ignorable – it does not appear L, so that the conservation law is

$$(m_1+m_2)\dot x + (m_2l\dot\phi\cos\phi) = \text{constant} \equiv p.$$

This conservation law can be used to eliminate $x(t)$ once p is specified and $\phi(t)$ is known. Thus we end up with a reduction to a single degree of freedom ϕ. As in our previous example, we can find the motion of ϕ by eliminating $\dot x$ from the energy in terms of $\phi(t)$ and the constant p and then using conservation of energy to find $\phi(t)$.

In the two body problem, we likewise used the conservation of total momentum – 3 conservation laws – to reduce the problem from 6 to 3 degrees of freedom. The change of variables from particle positions to center of mass and relative positions made manifest that the center of mass position is cyclic/ignorable. Unlike our two examples above, the particular geometry of center of mass and relative motion is such that this reduction is particularly simple: the two sets of degrees of freedom completely decouple. In general this decoupling need not occur, instead we get the scenario typified in the previous two examples. Indeed, we can use conservation laws to further reduce the relative motion problem, even though we don't get a decoupling.

Two Body Problem: Reduction to 1 radial degree of freedom.

We now return to our study of the 2 body problem, which has been reduced to the problem of a (fictitious) particle moving in a time independent central force field described by the Lagrangian

$$L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2} m \dot{\vec{r}}^2 - V(|\vec{r}|).$$

The EL equations are*

$$m\ddot{\vec{r}} = -\nabla V = -V'(|\vec{r}|)\frac{\vec{r}}{|\vec{r}|}.$$

To get this far we have used the conservation of total momentum. We still have conservation of angular momentum and conservation of energy at our disposal. In this section we implement the reduction procedure associated with conservation of angular momentum.

Note that $L(\vec{r}, \dot{\vec{r}})$ given above is rotationally invariant about any axis through the center of force (exercise). Thus the angular momentum,

$$\vec{l} = \vec{r} \times \vec{p},$$

is conserved. Note that this the angular momentum of the *relative* motion.

Homework Problem: Show that the conserved angular momentum $\vec{l}$ is the total angular momentum of the 2-body system defined relative to the center of mass.

Naively, since there are 3 conserved components in $\vec{l}$, it is tempting to believe we can use conservation of angular momentum alone to completely integrate the EL equations. This turns out not to be the case because of a technical subtlety: at any given point $\vec{r}$, only 2 components of angular momentum are linearly independent — this follows from the fact that

$$\vec{r} \cdot \vec{l} = 0.$$

For this reason we can use conservation of angular momentum to eliminate only 2 degrees of freedom. We could do this by writing out the equations and just computing things as we did in the Lagrangian reduction examples – those pendulum examples – we shall do this below. But, first, a more elegant and useful geometric point of view is worth exploring.

For any motion of the system, the vector $\vec{l}$ is fixed in space for all time. (Of course which vector this is depends upon the initial conditions, *i.e.,* upon which solution is being studied.) Because

$$\vec{r}(t) \cdot \vec{l} = 0 = \vec{p}(t) \cdot \vec{l},$$

it is easy to see that the motion $\vec{r}(t)$ of the "particle" is always in a plane orthogonal to $\vec{l}$, unless $\vec{l} = 0$. In the $\vec{l} = 0$ case there are three possibilities (1) $\vec{r} = 0$, which we don't allow;

* Notice that we allow division by the magnitude of $\vec{r}$. The configuration space for this "particle" is actually Euclidean space with a point – the origin – removed, $\mathbf{R}^3 - \{0\}$, since we will not consider the configuration where the two particles are at the same location.

(2) $\vec{p} = 0$, which can't happen for more than a single instant of time; the acceleration of $\vec{r}$ is such that at the next instant of time $\vec{p} \neq 0$ and is parallel to $\vec{r}$; (3) $\vec{r}$ and $\vec{p}$ are proportional, in which case the particles are moving along the line joining them. This is, of course, also motion in a plane. So, let us suppose, without loss of generality, that we have chosen a plane for the orbit to lie in. This specification amounts to a choice of the allowed initial conditions. Alternatively, the initial conditions pick out a plane orthogonal to $\vec{l}$ that the motion will lie within. For $\vec{l} \neq 0$, this choice is equivalent to specifying the *direction* of the angular momentum vector. Although we are to imagine having made a definite choice for this plane, it is completely arbitrary and no special features of the choice will be used in what follows. Furthermore, the rotational symmetry which spawned the conservation law means any other choice of plane, *i.e.,* another set of initial conditions, can be obtained from the one we choose by a rotation. Since rotations map solutions to solutions,* we can get all solutions from the one obtained by restricting to a given plane by simply rotating the obtained solutions. So, we can restrict our study to motion in a plane with no essential loss of generality. Thus we have used conservation of the angular momentum *direction* to reduce the system by one degree of freedom, *i.e.,* we now have a problem of motion of a "particle" in a plane.

As a good exercise you should confirm that (1) if the coordinates are chosen so that the initial position and velocity are in the x-y plane, $z(0) = 0, \dot{z}(0) = 0$, then $l_x = l_y = 0$; (2) the equations of motion preserve $z = 0 = \dot{z}$ for all time.

The equations of motion, when restricted to the x-y plane, can be expressed in polar coordinates (r, ϕ). The equations of motion for $r(t)$ and $\phi(t)$ are (exercise)

$$m\ddot{r} = mr\dot{\phi}^2 - V'(r),$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0.$$

These equations can be derived from the Lagrangian (exercise)

$$L(r, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

This Lagrangian can be obtained by simply setting $\theta = \frac{\pi}{2}$ in the original Lagrangian, $L(\vec{r}, \dot{\vec{r}})$, after expressing the Lagrangian in spherical coordinates (exercise). That this is the correct procedure in this particular step can be justified by a general theory of Lagrangian reduction, which we won't be able to explore. But it is worth noting that, in general, it is dangerous to simply eliminate degrees of freedom in the Lagrangian, since one can obtain incorrect EL equations by this procedure (see below). The reason it works

* Any symmetry of the Lagrangian transforms a solution of the EL equations into another solution of these same equations. It is a good exercise to prove this.

in this case is can be understood from the observation that we are permitted to impose the holonomic constraint $\theta = \pi/2$ in the Lagrangian, as discussed when analyzing dynamical systems with constraints. Anyway, we have now reduced our analysis to that of a system with 2 degrees of freedom, namely (r, ϕ).

Let us summarize the situation thus far. We started with 6 degrees of freedom. The conservation of total momentum allowed us to reduce to 3 (relative) degrees of freedom. The conservation of the *direction* of angular momentum allows us to reduce to 2 degrees of freedom. The reduced system at this stage consists of a "particle" with the reduced mass m moving in a plane (not including the origin) under the influence of a central force.

So far we have used the fact that the *direction* of $\vec{l}$ is a constant of motion, we still can use the fact that the magnitude of $\vec{l}$ is conserved. Having chosen our z-axis along $\vec{l}$, the magnitude of $\vec{l}$ is, of course, the z-component of angular momentum:

$$|\vec{l}| = mr^2\dot{\phi} = \text{constant} = l.$$

That this quantity *is* a constant of motion is easily seen directly from our latest Lagrangian, $L(r, \dot{r}, \dot{\phi})$, since ϕ is cyclic. As discussed in our previous examples, we can use this conservation law to reduce our analysis to that of a system with 1 degree of freedom. Specifically, we can use the conservation of l to eliminate $\phi(t)$ in terms of $r(t)$ and the constant l. We have

$$\dot{\phi} = \frac{l}{mr^2}, \quad \Longrightarrow \quad \phi(t) = \int_{t_0}^{t} ds\, \frac{l}{mr^2(s)}.$$

The reduced equation of motion for $r(t)$ is (exercise)

$$m\ddot{r} = -\frac{d}{dr}V_{\text{eff}}(r),$$

where

$$V_{\text{eff}} = V(r) + \frac{l^2}{2mr^2}.$$

This equation of motion can be derived from the Lagrangian for a system with a single degree of freedom (exercise):

$$L = \frac{1}{2}m\dot{r}^2 - V_{\text{eff}}(r).$$

Note that this Lagrangian is **not** *obtained by using conservation of angular momentum to algebraically eliminate $\dot{\phi}$ from $L(r, \dot{r}, \dot{\phi})$.* One way to see it is not permissible to eliminate $\dot{\phi}$ from the Lagrangian this way is that the formula being used to eliminate $\dot{\phi}$ is not compatible with the fixed endpoint conditions of the variational principle.

We now have reduced the problem to an autonomous system with one degree of freedom. As we have discussed, we can handle this problem using conservation of energy. The total conserved energy of the system *is* correctly given by simple substitution for $\dot\phi$ in the conserved energy for the motion in the plane (exercise):

$$E = \frac{1}{2}m\dot r^2 + V_{\rm eff}.$$

The contribution of the angular momentum to the effective potential energy represents an effective *repulsive* force (exercise) due to the angular motion of the "particle". For this reason, the term $\frac{l^2}{2mr^2}$ is often called the "centrifugal energy".

We have reduced the dynamics of the two body problem to that of a system of one degree of freedom, r, described by the Lagrangian

$$L = \frac{1}{2}m\dot r^2 - V_{\rm eff}(r),$$

where

$$V_{\rm eff} = V(r) + \frac{l^2}{2mr^2}.$$

Using the conservation of energy, in which

$$E = \frac{1}{2}m\dot r^2 + V_{\rm eff},$$

we can express the motion of the radial degree of freedom via (exercise)

$$t - t_0 = \pm\int_{r_0}^{r} \frac{1}{\sqrt{\frac{2}{m}(E - V(x)) - \frac{l^2}{m^2x^2}}}\, dx.$$

Thus using conservation of total momentum, angular momentum, and energy we are able to solve the equations of motion "up to quadrature".

Note that we tacitly assume that $E \neq V_{\rm eff}$ in the above formula. This excludes that case where the radial motion is at an equilibrium point: $\dot r \neq 0$. Of course when $\dot r = 0$ this is either a turning point, which is easily handled by continuity of the orbit (as discussed earlier), or one has $r =$ constant, *i.e.,* circular motion.

The orbits

The preceding reduction procedure provides a method for analyzing the motion in time of the two bodies. There is no closed form expression, however, for this motion – it is generally too complicated to be expressed using elementary analytic functions. It is possible, however, to find the *path* (relative) motion at the expense of dropping all information about how the path is traced out in time.

To begin, we have the relation between

$$\dot{\phi} = \frac{l}{mr^2(t)},$$

which means

$$d\phi = \frac{l}{mr^2(t)}dt.$$

Provided $l \neq 0$, we can parametrize the curves in configuration space with ϕ instead of t. Going back to the conservation of energy formula and trading dt for $d\phi$ we have (exercise)

$$d\phi = \pm \frac{l}{r^2\sqrt{2m(E - V(r)) - \frac{l^2}{r^2}}}\, dr. \qquad (1)$$

Thus we can express the angular motion as the integral:

$$\phi - \phi_0 = \pm \int_{r_0}^{r} \frac{l}{x^2\sqrt{2m(E - V(x)) - \frac{l^2}{x^2}}}\, dx. \qquad (2)$$

Typically, it is far easier to perform this integral than those needed to get the motion in time. Mathematically, this is because of the extra $\frac{1}{x^2}$ factor in the integrand. Physically this is because, by dropping the dependence on time, the integral does not have to convey so much information about the motion of the "particle".

We can understand this integral expression in another way. Consider the radial equation of motion:

$$m\ddot{r} = -V'_{\text{eff}}(r).$$

Make the change of variables $d\phi = \frac{l}{mr^2(t)}dt$. After a little algebra we get the differential equation for the orbit $r = r(\phi)$:

$$\frac{l}{r^2}\frac{d}{d\phi}\left(\frac{l}{mr^2}\frac{dr}{d\phi}\right) = -V'_{\text{eff}}(r).$$

The solution $r = r(\phi)$ is determined by the integral expression (2) given above. This follows from the first integral for the orbit equation

$$\frac{1}{2}\frac{l^2}{m^2r^4}r'^2 + V_{\text{eff}} = E,$$

which is just conservation of energy expressed in terms of the orbit $r(\phi)$.

The relation between r and ϕ from the last integral gives the path traced out by the "particle" via $\phi = \phi(r)$. If desired, we can invert to get $r = r(\phi)$, which is the *orbit* of the particle. Knowledge of the orbit (path) carries no information about how the path is traced out in time. This is obtained from the integral relating r and t, giving $r = r(t)$ and $\phi = \phi(t) = \phi(r(t))$. For a given choice of the energy E, the values of r where

$$E = V_{\text{eff}}(r),$$

define the turning points of the radial motion. At the turning points the radial velocity vanishes; it changes sign as the particle passes through the turning point. Of course, the "particle" doesn't actually come to rest at the turning point since its angular motion is monotonic in time, as can be seen from the conservation of angular momentum formula (exercise). If E is such that there are two turning points then we have a bound state and the motion is confined to an annular region in the plane of motion; we have orbital motion. If there is only one turning point the motion is unbound and we have a scattering situation, which we may discuss later.

For now, let us restrict our attention to bound motion. If the potential energy function is such that V_{eff} has a minimum, and if E takes that minimum value, then the radial motion is that of stable equilibrium. In this case $r =$ constant and the motion is circular and, of course, periodic. More generally, however, the motion is considerably more complicated. Even though the bound motion is periodic in the radial variable, this does not mean that the motion in the plane is periodic. This is because the time it takes for ϕ to change by 2π need not be the same as the time it takes for r to pass through one cycle. In detail, during the time that the radial variable passes through one cycle, the angle changes by (exercise)

$$\Delta\phi = 2\int_{r_0}^{r_1} \frac{l}{x^2\sqrt{2m(E - U(x)) - \frac{l^2}{x^2}}}\,dx,$$

where r_0 and r_1 are the turning points of the radial motion. For the path of the "particle" in the plane to define a closed orbit $\Delta\phi$ must be a rational multiple of 2π:

$$\Delta\phi = \frac{p}{q}(2\pi),$$

so that after q cycles in the radial motion the angular variable will have made a net change of $2\pi p$, *i.e.,* the "particle" will have made p revolutions.

Generically, that is, for "almost all" potentials $V(r)$, the condition just stated for closure of the orbit will *not* be satisfied. Thus, for a typical central force problem the relative motion will correspond to that of a "particle" that follows a curve about the origin which is bounded by r_0 and r_1 and eventually passes through every point in this annular region. Exceptions to this occur when $V \propto \frac{1}{r}$ (Kepler problem; to be discussed shortly) or $V \propto r^2$ (isotropic oscillator). We shall see shortly that there is a hidden symmetry and conservation law in these exceptional cases that "explains" the closure of the orbits.

Finally we note that thanks to the repulsive effect of the "centrifugal energy", the "particle" cannot reach the center of force $r = 0$ (where the 2 bodies collide) unless the central force is sufficiently attractive — and singular — as $r \to 0$. To see this, simply consider the formula for the energy. Assuming that r is changing, and using the fact that the radial contribution to the kinetic energy is positive, we have the inequality (exercise)

$$r^2 V(r) + \frac{l^2}{2m} < Er^2$$

or

$$r^2 (V(r) - E) < -\frac{l^2}{2m}.$$

This means that as r approaches zero $V \sim -cr^{-\alpha}$, up to an additive constant, with $c > 0$ appropriately chosen, and $\alpha \geq 2$. Of course, if $l = 0$ this condition has much weaker implications for the potential.

Exercise: What is the nature of the motion if $l = 0$?