

Physics 6010, Fall 2016

Symmetries and Conservation Laws

Relevant Sections in Text: §2.6, 2.7

Symmetries and Conservation Laws

By a *conservation law* we mean a quantity constructed from the coordinates, velocities, accelerations, *etc.* of the system that does not change as the system evolves in time. When the equations of motion are second order, conservation laws typically arise as functions on the velocity phase. Although I won't be too particular about it, one often restricts the terminology "conservation law" or "first integral" to denote quantities built from the phase space variables only - not explicitly involving time. The term "constant of the motion" is reserved for the more general class of quantities which may include time in their definition. For example, a free particle moving in one dimension, say x , has its velocity \dot{x} as a conserved quantity (also a constant of the motion) – this is the conservation of momentum, of course. A constant of the motion (but not a conservation law) is provided by the explicitly time-dependent quantity $x_0 = x - \dot{x}t$, as you can easily check. This quantity physically represents the position at time $t = 0$.

For a more detailed example, consider a harmonic oscillator in one dimension. Let the generalized coordinate be x . The Lagrangian is

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

where m, k are constants. The equation of motion is of the form

$$m\ddot{x} + kx = 0.$$

The *energy* is given by

$$E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

The energy is conserved because we have the identity

$$\frac{dE}{dt} = \dot{x}(m\ddot{x} + kx),$$

so that, when E is evaluated on a solution $x = x(t)$ to the equations of motion, the resulting function $E(t) \equiv E(x(t), \frac{dx(t)}{dt})$ satisfies $\frac{dE(t)}{dt} = 0$. To see this explicitly, recall the general solution to the equation of motion is

$$x(t) = A \cos(\omega t + \alpha),$$

where A and α are constants (determined, *e.g.*, by initial conditions) and $\omega = \sqrt{\frac{k}{m}}$. We then have

$$E(x(t), \frac{dx(t)}{dt}) = \frac{1}{2}kA^2,$$

which is indeed time independent.

I do not think I need to impress upon you the importance of conservation laws in physics. On the practical side, one can use conservation of energy, momentum, *etc.* to unravel many aspects of the motion of a system without having to explicitly integrate the equations of motion. Indeed for systems with one degree of freedoms, a conservation law usually determines everything! More generally, if there are enough conservation laws it is possible to completely solve for the motion. On the other hand, I cannot emphasize enough the fact that almost all dynamical systems are not simple enough for us to study their motion by integrating the equation of motion, *i.e.*, there are usually *not* enough conservation laws to completely determine the motion. Still, even in these cases, the conservation laws provide some of the principal clues we have as to the dynamical behavior of such systems. At a deeper level, we use conservation laws to guide us in our quest to find what are the physical laws governing the universe. Throughout the history of physics we have repeatedly revised our formulation of the laws of nature. The current state of the art involves the “standard model” of strong and electroweak forces along with Einstein’s general theory of relativity for the gravitational force. In building these theories, the myriad of conservation laws observed in nature* form the foundation for the work of the theoretical physicist. It is reasonable to suppose that future generations of physicists will further revise our theory of matter and interactions of matter, but it much less likely that these theories will not incorporate conservation laws.

One of the principal advantages of the Lagrangian formulation of mechanics (and its field theoretic generalizations) is the power it provides for analyzing the existence of conservation laws. The fundamental result we want to explore now is that *the existence of conservation laws stems from the existence of symmetries of the Lagrangian*.

Translation Symmetry and Conservation of Momentum.

Let us begin by noting a very easy result: when a (generalized) coordinate does not appear in the Lagrangian, then a conserved quantity results. When a coordinate, q^1 say, is absent in the Lagrangian we say that q^1 is *cyclic* or *ignorable*. In this case we have

$$\frac{\partial L}{\partial q^1} = 0.$$

* Of course there is energy, momentum and angular momentum, but recall we also have conservation of charge, baryon number, lepton number, weak isospin, strangeness, charm, *etc.*

The EL equation for the q^1 degree of freedom is then simply (exercise)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^1} = 0,$$

which says that the quantity $\frac{\partial L}{\partial \dot{q}^1}$ is conserved. For a Newtonian particle for which the q^i are the Cartesian coordinates of the particle, the conserved quantity resulting from a cyclic coordinate is just the corresponding component of the usual Newtonian momentum (exercise). This motivates the following definition. For a given degree of freedom, q^i , the quantity

$$p_i(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}^i}$$

is called the *canonical momentum conjugate to q^i* . We see that if q^i is cyclic, its conjugate momentum is conserved. Note that if we define \mathcal{E}_i as the EL expressions,

$$\mathcal{E}_i(q, \dot{q}, \ddot{q}, t) := \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i},$$

then the conservation law associated with cyclic q^1 follows from the identity

$$\frac{d}{dt} p_1 = \mathcal{E}_1.$$

As a simple example of this, consider projectile motion: a particle with mass m moves (for simplicity) in two dimensions with configuration space coordinates $q^i = (x, y)$ and a potential energy function $V(x, y) = mgy$. The Lagrangian is, of course,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy,$$

and x is clearly cyclic. The EL equations reveal the conservation law – the x -component of momentum is conserved:

$$\frac{d}{dt}(m\dot{x}) = 0, \quad \frac{d}{dt}(m\dot{y}) = -mg.$$

Exercise: Suppose you were a bit naive and used an x - y coordinate system rotated by an angle α relative to the one used just above. What happens to the cyclic coordinate and its conservation law?

If a coordinate, say, q^1 , is cyclic, this obviously implies that the Lagrangian is invariant under translations of this coordinate (because the Lagrangian doesn't depend upon that coordinate!):

$$L(q^1 + \text{constant}, q^2, \dots, \dot{q}^1, \dot{q}^2, \dots, t) = L(q^1, q^2, \dots, \dot{q}^1, \dot{q}^2, \dots, t).$$

Thus translation in the q^1 coordinate does not change the form of the Lagrangian. We say that translation in q^1 is a *symmetry* of the Lagrangian, or that the Lagrangian is *invariant* under translations in q^1 . So, invariance of the Lagrangian with respect to translations in the q^i coordinate leads to conservation of the canonical momentum p_i . Note that the EL equations will not depend upon q^1 either, so that they are the exact same equations no matter what values q^1 takes — the equations are invariant under translations in q^1 . We say that translations in q^1 are a symmetry of the EL equations. But it is the symmetry of the Lagrangian which leads to the conservation law.*

As mentioned above, the choice of the term “momentum” to describe p_i stems from the fact that for Lagrangian for a system of Newtonian particles:

$$L = \frac{1}{2}m_{(1)}\dot{\vec{r}}_{(1)}^2 + \frac{1}{2}m_{(2)}\dot{\vec{r}}_{(2)}^2 + \dots - V(\vec{r}_{(1)}, \vec{r}_{(2)}, \dots, t),$$

we have (exercise)

$$\vec{p}_{(i)} = m_{(i)}\dot{\vec{r}}_{(i)}.$$

We will call $m_{(i)}\dot{\vec{r}}_{(i)}$ the *mechanical momentum* of the i^{th} particle. For system of particles moving under the influence of a potential via Newton’s second law, the Cartesian components of the canonical momenta are the same as that of the mechanical momenta. If the Lagrangian is invariant under a translation of one of the coordinates, this means the potential energy is likewise invariant, and conversely (exercise). It is clear why this leads to a conservation law from the Newtonian point of view: if the potential is translationally invariant in a certain direction, then its derivative in that direction – which gives the force in that direction – vanishes, leaving the corresponding momentum component unchanged in time.

When we use generalized coordinates to define the configuration of the system it is possible that the canonical momentum is not what you would usually call the mechanical momentum. Moreover, one may have translational symmetries which do not manifest themselves via cyclic coordinates. For example, consider a particle moving in 3-d under the influence of a *central force* derivable from the potential energy $V = V(|\vec{r}|)$. Because of the spherical symmetry of the problem, it is natural to use spherical polar coordinates; the Lagrangian is (exercise)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r).$$

Clearly the coordinate ϕ is cyclic. The resulting conservation law is

$$\frac{d}{dt}p_\phi = 0,$$

* In general, every symmetry of the Lagrangian is a symmetry of the equations of motion. But the converse is not true.

where

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi}.$$

In this case the conserved quantity is the z -component of *angular momentum*. More on this shortly.

It is instructive to consider the special case $V = 0$ in spherical polar coordinates. Of course, in Cartesian coordinates the Lagrangian is translationally invariant in each of (x, y, z) . That translational symmetry still exists in spherical polar coordinates, it is just not apparent. We will see how to think about this issue shortly.

Another important case where the mechanical and canonical momenta differ—even for Cartesian coordinates—occurs for particle motion in a prescribed electromagnetic field. Recall that the Lagrangian is

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - q\phi + \frac{q}{c}\vec{A} \cdot \dot{\vec{r}}.$$

The canonical momentum is, in vector notation, (exercise)

$$\vec{p}(\vec{r}, \dot{\vec{r}}, t) = m\dot{\vec{r}} + \frac{q}{c}\vec{A}(\vec{r}, t).$$

The canonical momentum in this case need not have immediate physical significance since it depends upon the choice of vector potential, which is not uniquely determined by a given electromagnetic field. Still, it is the *canonical* momentum which will be conserved when a coordinate is cyclic.*

To summarize, if a coordinate does not appear in the Lagrangian, *i.e.*, the Lagrangian admits a translational symmetry in a coordinate, its conjugate momentum will be conserved.

Time translation Symmetry and Energy Conservation

We consider a system with generalized coordinates q^i and Lagrangian

$$L = L(q, \dot{q}, t).$$

We assume that L is unchanged by a time translation:

$$L = L(q, \dot{q}) \quad \Longleftrightarrow \quad \frac{\partial L}{\partial t} = 0.$$

This lack of *explicit* t dependence means the Lagrangian is the same function on the velocity phase space for all time. This also implies the equations of motion are the same differential

* Also, it turns out that it is the canonical momentum which is to be represented by a derivative operator in the position representation of quantum mechanics.

equations for all time t (exercise).^{*} Thus we say that such systems admit a *time translation symmetry*.

Now consider the restriction of the Lagrangian to a curve $q^i(t)$ satisfying the EL equations. Denote this restriction by $L(t)$:

$$L(t) := L(q(t), \frac{dq(t)}{dt}).$$

The dependence of L on time comes solely through its dependence on $q^i(t)$ and $\dot{q}^i(t)$. Thus, on such a curve, (exercise)

$$\frac{dL(t)}{dt} = \frac{\partial L}{\partial q^i} \dot{q}^i + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i,$$

where it is understood that all quantities on the right hand side of the equation are evaluated on the curve. On a curve satisfying the EL equations we have

$$\frac{\partial L}{\partial \dot{q}^i} = \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}^i}.$$

So, assuming the curve $q^i(t)$ satisfies the EL equations, we have

$$\begin{aligned} \frac{dL}{dt} &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) \dot{q}^i + \frac{\partial L}{\partial \ddot{q}^i} \ddot{q}^i \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i \right). \end{aligned}$$

We conclude that, when evaluated on solutions $q^i(t)$ of the EL equations, we have the result:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \right) = 0.$$

Evidently, for time translation invariant Lagrangians the quantity $p_i \dot{q}^i - L$ is conserved. To see what kind of quantity is conserved here, let us go back to the harmonic oscillator,

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2.$$

This Lagrangian is time translation invariant. You can easily check that

$$\frac{\partial L}{\partial \dot{x}} \dot{x} - L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2,$$

which is the conserved energy of the oscillator. This leads us to define the *canonical energy* $E(q, \dot{q}, t)$ of a system described by the Lagrangian $L(q, \dot{q}, t)$ as

$$E(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L.$$

^{*} Such equations, with no explicit dependence on the independent variable, are called *autonomous*.

Our result is that, when $\frac{\partial L}{\partial t} = 0$, and when evaluated on a curve satisfying the equations of motion, the canonical energy is conserved:

$$\frac{d}{dt}E(q(t), \dot{q}(t)) = 0.$$

We can express energy conservation as an identity which holds when the Lagrangian does not depend upon time. For a general Lagrangian $L = L(q, \dot{q}, t)$ we have (exercise)

$$\frac{d}{dt}E = -\dot{q}^i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial t}.$$

From this identity you can see that E , when viewed as a function on velocity phase space is unchanged as you move along a curve satisfying the EL equations provided the Lagrangian has no explicit time dependence.

As already mentioned, the use of the term “energy” to label this conservation law stems from the fact that for the usual type of Newtonian system the quantity $E(q, \dot{q}, t)$ corresponds to our familiar definition of *mechanical energy* – kinetic plus potential. For example, a particle moving in a given potential has the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}, t).$$

We have – using Cartesian coordinates (exercise)

$$\frac{\partial L}{\partial \dot{x}^i} \dot{x}^i = m\dot{x}_i \dot{x}^i = 2T,$$

so that

$$E = 2T - (T - V) = T + V.$$

Thus we recover the usual definition of energy as the sum of kinetic and potential energies. We learn, then, that this quantity is conserved for a particle moving in a given potential provided the potential is time independent.

More generally, whenever $L = T - V$ and the kinetic energy $T(q, \dot{q}, t)$ is a homogeneous function of degree two in the velocities, we have that (exercise)

$$\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i = 2T,$$

and hence,

$$E = T + V.$$

It should be noted that, unless the above requirements are satisfied, the canonical energy – the conserved quantity associated with time translation invariance of a Lagrangian – need

not be the *mechanical energy* $T + V$. In particular, it is possible that the canonical energy *is* conserved, but is not the mechanical energy. Conversely, the mechanical energy might be conserved, but need not in general be the same as the canonical energy.

Rotational Symmetry and Conservation of Angular Momentum

Here we demonstrate that rotational symmetry of a Lagrangian leads to conservation of angular momentum. You have already seen an example of this: the particle moving in a central force has the spherical coordinate ϕ as a cyclic variable. The conjugate momentum that is conserved is the z component of angular momentum. The kinetic energy is invariant under rotations about any axis; for a central force the potential energy $V = V(r)$ and hence the Lagrangian $L = T - V$ is invariant under rotations about *any* axis. This implies that we can choose the z -axis along any direction and the corresponding component of angular momentum will be conserved. Thus *all* components of angular momentum will be conserved for a particle moving in a central force. More generally a Lagrangian which is rotationally invariant about some axis will have the total angular momentum along that axis conserved. In what follows we show this directly.

Let us begin with the Lagrangian for a single particle, $L(\vec{r}, \dot{\vec{r}}, t)$. We want to impose the condition that the Lagrangian is rotationally invariant in order to see the consequences. Thus we need to get a mathematical handle on how the position and velocity of a particle change under rotations. Recall that to specify a rotation one needs to pick an axis of rotation and an angle. Given these data, one can write down formulas for how vectors transform. For simplicity, we use a very important strategy: focus on *infinitesimal rotations*. The idea is that finite rotations can be built up by “many” infinitesimal transformations. In particular, a Lagrangian is invariant under rotations about some axis if and only if it is invariant about infinitesimal rotations about that axis.

We consider the change δL in the Lagrangian produced by an *infinitesimal* rotation around an axis defined by the unit vector \vec{n} by an angle $\epsilon \ll 1$. First, we point out that under such an infinitesimal rotation we have

$$\delta \vec{r} = \epsilon \vec{n} \times \vec{r},$$

and

$$\delta \dot{\vec{r}} = \epsilon \vec{n} \times \dot{\vec{r}}.$$

This is easily verified with a judicious choice of coordinates, which you should verify as an exercise. (Choose your z -axis along \vec{n} , compute the effect of a rotation around by ϵ on \vec{r} or $\dot{\vec{r}}$ and expand everything to first order in ϵ . You can also convince yourself of the validity of the above formulas by drawing some pictures.) Anyway, under a rotation about \vec{n} by

$\epsilon \ll 1$ we have the change in the Lagrangian to first order in ϵ given by (exercise)

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial x^i} \delta x^i + \frac{\partial L}{\partial \dot{x}^i} \delta \dot{x}^i \\ &= \frac{\partial L}{\partial x^i} \epsilon (\vec{n} \times \vec{r})^i + \frac{\partial L}{\partial \dot{x}^i} \epsilon (\vec{n} \times \dot{\vec{r}})^i.\end{aligned}$$

Here we have set $x^i = (x, y, z)$.

Suppose the Lagrangian is rotationally invariant about \vec{n} . An example of this would be the Lagrangian for a charged particle in a uniform electric field oriented in the direction \hat{n} :

$$L = \frac{1}{2}mv^2 + qE \hat{n} \cdot \vec{r}, \quad E = \text{const.}$$

This means that δL given above must vanish. So we know the following identity holds for such a rotationally invariant Lagrangian:

$$\frac{\partial L}{\partial x^i} (\vec{n} \times \vec{r})^i + \frac{\partial L}{\partial \dot{x}^i} (\vec{n} \times \dot{\vec{r}})^i = 0.$$

In terms of the canonical momentum \vec{p} conjugate to \vec{r} , the EL equations are

$$\mathcal{E}_i \equiv \frac{\partial L}{\partial x^i} - \dot{p}_i = 0.$$

The rotational symmetry identity, written above, can be expressed in terms of the EL equations as

$$(\dot{\vec{p}} + \vec{\mathcal{E}}) \cdot (\vec{n} \times \vec{r}) + \vec{p} \cdot (\vec{n} \times \dot{\vec{r}}) = 0.$$

This identity can be rewritten as (exercise)

$$\frac{d}{dt} [\vec{n} \cdot (\vec{r} \times \vec{p})] = -(\vec{n} \times \vec{r}) \cdot \vec{\mathcal{E}}.$$

This is the statement that the component along \vec{n} of the angular momentum,*

$$\vec{M} = \vec{r} \times \vec{p},$$

is conserved when evaluated on solutions to the EL equations. If the rotational invariance is valid for any choice of \vec{n} then all components of \vec{M} will be conserved. Note however that it is the *canonical* momentum that features in this conservation law, not the mechanical momentum. We thus call \vec{M} the *canonical angular momentum* and we call $\vec{r} \times m\vec{v}$ the *mechanical angular momentum*.

* I will use the symbol \vec{M} for angular momentum since it is too easy to confuse the more traditional notation \vec{L} with the Lagrangian.

Here are some elementary examples of rotational symmetry and conservation of angular momentum. A particle moving in a central force field has a Lagrangian:

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - U(|\vec{r}|).$$

This Lagrangian is clearly invariant under rotations. As a really good exercise you should verify the infinitesimal rotational invariance identity:

$$\frac{\partial L}{\partial x^i}(\vec{n} \times \vec{r})^i + \frac{\partial L}{\partial \dot{x}^i}(\vec{n} \times \dot{\vec{r}})^i = 0.$$

Thus all components of angular momentum are conserved. This can, of course, be checked directly by using the equations of motion. Using the fact that the particle obeys Newton's second law with a force directed along the position vector \vec{r} , you can easily prove that $\vec{r} \times \vec{p}$ is conserved (exercise).

As another example, consider a particle moving in a uniform force field \vec{F} . This means that the Cartesian components of \vec{F} are constants. A Lagrangian for this system is (exercise)

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + \vec{F} \cdot \vec{r}.$$

This Lagrangian would, *e.g.*, describe the motion of a particle near the earth's surface. While the kinetic energy is rotationally invariant about any axis, the potential energy in this example is *not* invariant under all rotations. This is because the vector \vec{F} is fixed once and for all; \vec{F} is not allowed to rotate along with \vec{r} and $\dot{\vec{r}}$. Another way to see this is to suppose that we choose our z-axis along \vec{F} . Then the potential energy is of the form $U = -Fz$. Under a rotation about anything but the z-axis this function will change! The potential energy *is* invariant under rotations about an axis parallel to \vec{F} (exercise). Another good exercise: verify

$$\frac{\partial L}{\partial x^i}(\vec{F} \times \vec{r})^i + \frac{\partial L}{\partial \dot{x}^i}(\vec{F} \times \dot{\vec{r}})^i = 0.$$

for this Lagrangian. Consequently the component of angular momentum along \vec{F} will be conserved. Again, using the equations of motion (EL equations),

$$m\ddot{\vec{r}} = \vec{F},$$

you can verify the conservation law explicitly (exercise).

Spacetime Symmetry, Closed Systems, and the Galileo Group

The usual model of space and time, which is tacitly part of Newtonian mechanics, assumes that space and time are homogeneous and that space is isotropic. Thus a closed

system (isolated from the external world) should “behave” the same – obey the same dynamical laws – no matter where in the universe it is located, when the system is studied, and no matter how the system is oriented in space. This suggests that the Lagrangian for such a system should be unchanged under space and time translations of the whole system as well as rotations of the whole system. Given the connection between symmetries and conservation laws we then expect corresponding conservation laws. We explore this briefly here.

Let us consider a closed — isolated — system of interacting particles labeled by positions, $\vec{r}_{(1)}$, $\vec{r}_{(2)}$, *etc.* Think of the particles as living inside a “black box” which does not interact with the outside world. The assumption of homogeneity of time implies that the Lagrangian for the system does not have any explicit time dependence and the total canonical energy is conserved, as discussed previously. By the assumption of the homogeneity of space the physical system behaves the same no matter where it is located in the universe. The Lagrangian will be unchanged under a spatial translation of the entire system. This means that the Lagrangian is invariant under the transformation

$$\vec{r}_{(a)} \longrightarrow \vec{r}_{(a)} + \vec{b}, \quad \forall i$$

where \vec{b} is any constant vector. Labeling the *Cartesian* coordinates as $x_{(a)}^i$, $a = 1, 2, \dots$, *i.e.*, $x_{(1)}^i = (x_1, y_1, z_1)$, *etc.* this implies (exercise)

$$\sum_{a,i} b^i \frac{\partial L}{\partial x_{(a)}^i} = 0.$$

The EL equations for the motion of the particles imply (exercise)

$$\frac{d}{dt} \sum_{a,i} b^i p_{i(a)} \equiv \frac{d}{dt} \sum_a \vec{b} \cdot \vec{p}_{(a)} = 0,$$

where

$$p_{i(a)} = \frac{\partial L}{\partial \dot{x}_{(a)}^i},$$

and we have denoted the vector canonical momentum for each particle by $\vec{p}_{(a)}$. The quantity

$$\vec{P} = \sum_a \vec{p}_{(a)}$$

represents the *total linear momentum* for the system. We see that the invariance of physical laws under a space translation by \vec{b} implies that the component of the total momentum along \vec{b} is conserved. Of course, since \vec{b} is arbitrary, all 3 components of \vec{P} are conserved for a closed system because of the homogeneity of space.

For a Newtonian system, the Lagrangian is thus of the form

$$L = \sum_a \frac{1}{2} m_{(a)} \dot{\vec{r}}_{(a)}^2 - V(\vec{r}_{(1)}, \vec{r}_{(2)}, \dots).$$

The conserved total energy is of the form

$$E = \sum_{a,i} \frac{\partial L}{\partial \dot{x}_{(a)}^i} \dot{x}_{(a)}^i - L = \sum_a \frac{1}{2} m_{(a)} \dot{\vec{r}}_{(a)}^2 + V(\vec{r}_{(1)}, \vec{r}_{(2)}, \dots).$$

The conserved total momentum is

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{r}}_{(a)}} = \sum_a m_{(a)} \dot{\vec{r}}_{(a)}.$$

Although I won't prove it here, it is worth noting that the requirement of symmetry of the Lagrangian under spatial translations of the system implies that the potential energy function can only depend upon the position vectors through their pairwise vector differences, *i.e.*,

$$V = V(\vec{r}_{(1)} - \vec{r}_{(2)}, \vec{r}_{(1)} - \vec{r}_{(3)}, \dots).$$

As an example, let us consider the Earth-Sun system ignoring all other external interactions. The Lagrangian is of the form

$$L = \frac{1}{2} m_E \dot{\vec{r}}_E^2 + \frac{1}{2} m_S \dot{\vec{r}}_S^2 - V(|\vec{r}_E - \vec{r}_S|).$$

Here we usually take the potential energy to be of the form

$$V = \frac{G m_E m_S}{|\vec{r}_E - \vec{r}_S|}.$$

But more sophisticated choices for V are possible. In any case, the homogeneity of space demands that $V = V(|\vec{r}_E - \vec{r}_S|)$, so the Lagrangian is invariant under any translation of the form

$$\vec{r}_E \rightarrow \vec{r}_E + \vec{b}, \quad \vec{r}_S \rightarrow \vec{r}_S + \vec{b}.$$

Neither the momentum of the Earth nor that of the Sun is conserved. But, because the Lagrangian is translationally invariant, the total momentum:

$$\vec{P} = m_E \dot{\vec{r}}_E + m_S \dot{\vec{r}}_S$$

is conserved.

Let us now consider the implications of spatial isotropy. The isotropy of space implies that a closed system will have a Lagrangian which is invariant under rigid rotations of all the particles in the system. We have particles labeled by positions $\vec{r}_{(a)}$ and velocities $\dot{\vec{r}}_{(a)}$.

Suppose the system admits a rotationally invariant Lagrangian. More precisely, suppose the Lagrangian is assumed invariant under a simultaneous rotation of all the positions and velocities about an axis \vec{n} (through the origin). The infinitesimal invariance condition generalizes in a straightforward manner (exercise):

$$\sum_a \left(\frac{\partial L}{\partial \vec{r}_{(a)}} \cdot (\vec{n} \times \vec{r}_{(a)}) + \vec{p}_{(a)} \cdot (\vec{n} \times \dot{\vec{r}}_{(a)}) \right) = 0.$$

As before, we can restrict this to a curve satisfying the EL equations to find (exercise)

$$\sum_a \left(\dot{\vec{p}}_{(a)} \cdot (\vec{n} \times \vec{r}_{(a)}) + \vec{p}_{(a)} \cdot (\vec{n} \times \dot{\vec{r}}_{(a)}) \right) = 0.$$

As before we can rearrange this as (exercise)

$$\frac{d}{dt} \sum_a [\vec{n} \cdot (\vec{r}_{(a)} \times \vec{p}_{(a)})] = \frac{d}{dt} \left(\vec{n} \cdot \sum_a \vec{M}_a \right) = 0.$$

Thus invariance of the Lagrangian under a simultaneous rotation about the axis \vec{n} of all degrees of freedom leads to conservation of the component along \vec{n} of the *total angular momentum*,

$$\vec{M}_{total} = \sum_a \vec{M}_a.$$

As an example, let us return yet again to the Earth-Sun system (2-body central force problem). The Lagrangian is of the form

$$L = \frac{1}{2} m_E \dot{\vec{r}}_E^2 + \frac{1}{2} m_S \dot{\vec{r}}_S^2 - V(|\vec{r}_E - \vec{r}_S|).$$

Because the scalars v_E , v_S , and $|\vec{r}_E - \vec{r}_S|$ are invariant under rotations, the Lagrangian is invariant under any simultaneous rotation of the Earth and Sun variables (exercise). The conserved angular momentum is (exercise)

$$\vec{P} = m_E \vec{r}_E \times \dot{\vec{r}}_E + m_S \vec{r}_S \times \dot{\vec{r}}_S.$$

To summarize, closed systems can always be expected to have conservation laws for the total energy, total momentum, and total angular momentum by virtue of the homogeneity and isotropy of space and time.

I should emphasize that it is possible to have conservation laws for open systems as well. We have already seen examples, but let me revisit this idea. For example, the Earth-Sun system we have discussed is a closed system admitting the usual conservation laws. But we often approximate the motion of the system by assuming the sun is fixed in space,

letting the dynamical system consist of the motion of the Earth in the fixed potential of Sun. So, if we simplify our model by assuming we can choose a reference frame in which the Sun is fixed, *e.g.*, $\vec{r}_{Sun} = 0$, then the resulting system (Earth in a central force field due to the “environment” – the Sun) is not a closed system and need not have all the space and time symmetries. In this case the system lacks space translation invariance since it does matter to the earth where it is in space (relative to the fixed sun). Consequently, the Earth’s linear momentum is not conserved. But the Lagrangian still has no explicit time dependence and is still rotationally invariant (about axes going through the sun) and so the canonical energy,

$$E = \frac{1}{2}m_E\dot{r}_E^2 - \frac{Gm_Em_S}{|\vec{r}_E|},$$

is conserved, as is the angular momentum,

$$\vec{M} = m_E\dot{r}_E \times \vec{r}_E.$$

Of course, both of these conservation laws are used to great effect in studying the motion of the Earth.

Noether’s Theorem

We have seen that the familiar conservation laws for energy, momentum, and angular momentum all follow from symmetry properties of the Lagrangian. In your homework you had to exhibit additional conservation laws in nature. It is natural to ask whether other—indeed, if all—conservation laws arise via symmetries of a Lagrangian. The answer is yes. This result, which is a key reason for the power of the Lagrangian formalism, is known as *Noether’s theorem*. While we will not prove Noether’s theorem here, we can give a good picture of how it works. Here it is.

Consider an infinitesimal transformation

$$\delta q^i = F^i(q, \dot{q}, t), \quad \delta \dot{q}^i = \frac{d}{dt}F^i.$$

The idea here is that, given a curve $q^i = q^i(t)$ the infinitesimal transformation defines a varied curve

$$q^i(t) + \delta q^i(t) = q^i(t) + F^i(q(t), \frac{dq(t)}{dt}, t).$$

Note however that the variation here does *not* need to vanish at any endpoints. The change in the Lagrangian (on a curve) due to this infinitesimal transformation can be computed via

$$\delta L = \frac{\partial L}{\partial q^i} F^i + \frac{\partial L}{\partial \dot{q}^i} \frac{d}{dt} F^i.$$

We say the Lagrangian is *invariant* under this transformation — equivalently: $\delta q = F$ defines an *infinitesimal symmetry* of the Lagrangian — if there exists a function G on the velocity phase space such that

$$\delta L = \frac{dG}{dt}.$$

Because two Lagrangians differing by a time derivative give the same equations of motion, this is a suitable notion of symmetry. (Note that G may vanish in many cases.)

Noether's theorem then guarantees that

$$Q = \frac{\partial L}{\partial \dot{q}^i} F^i - G$$

is conserved when the EL equations hold:

$$\mathcal{E}_i \equiv \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0.$$

To see this is a straightforward computation. On the one hand we have for any variation (exercise)

$$\delta L = \mathcal{E}_i \delta q^i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right).$$

For a variation which is an infinitesimal symmetry we have

$$\delta L = \frac{dG}{dt}.$$

Putting these two results together we have for an infinitesimal symmetry $\delta q^i = F^i$:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} F^i - G \right) = -\mathcal{E}_i F^i.$$

Thus, for solutions of the EL equations Q is conserved.

Noether's Theorem (informally stated):

There is a one to one correspondence between Lagrangian symmetries and constants of the motion. Given a Lagrangian symmetry, δq^i (which may involve a function G in a total derivative), the corresponding constant of the motion is

$$Q = \frac{\partial L}{\partial \dot{q}^i} \delta q^i - G.$$

Remark:

Note that discrete symmetries (*e.g.*, reflections) will not give rise to conservation laws since they do not have an infinitesimal form.

As an example of the use of this theorem, let us reconsider conservation of momentum and energy. Consider a Lagrangian L with q^1 cyclic:

$$\frac{\partial L}{\partial q^1} = 0.$$

Consider the following transformation:

$$q^i(\alpha) = q^i + \alpha \delta^{i1}.$$

This is a translation of q^1 . The infinitesimal transformation is (exercise)

$$\delta q^i = \delta^{i1}.$$

In this case the total derivative term does not appear, *i.e.*, $G = 0$:

$$\delta L = 0.$$

We get (exercise)

$$\begin{aligned} Q &= \sum_j \frac{\partial L}{\partial \dot{q}^j} \delta^{1j} \\ &= p_1. \end{aligned}$$

Next consider conservation of energy. For conservation of energy to hold we must assume that $\frac{\partial L}{\partial t} = 0$. The infinitesimal symmetry is

$$\delta q^i = \dot{q}^i.$$

This transformation can be viewed as a time translation since on a curve $q^i(t)$, infinitesimally, (exercise)

$$q^i(t + \epsilon) \approx q^i(t) + \epsilon \frac{dq(t)}{dt}.$$

That this *is* a Lagrangian symmetry when $\frac{\partial L}{\partial t} = 0$ is easily seen:

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial q^i} \dot{q}^i + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i \\ &= \frac{dL}{dt}. \end{aligned}$$

The last equality follows from the assumption that the Lagrangian has no explicit time dependence. We conclude that $\delta q^i = \dot{q}^i$ is a Lagrangian symmetry in which the total derivative function is given by $G = L$. Applying Noether's theorem we get (exercise)

$$Q = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L = E.$$

As an exercise you can use Noether's theorem to re-derive the relation between rotational symmetry and conservation of angular momentum.

Finally, we remark we do not need to have (explicitly) a cyclic coordinate to get a conservation law. We just need an infinitesimal transformation for which the Lagrangian only changes by a total time derivative. For example, consider a particle moving in a central force field:

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - U(|\vec{r}|).$$

Working in Cartesian coordinates, all 3 coordinates appear in U :

$$U = U(\sqrt{x^2 + y^2 + z^2}).$$

However, the following infinitesimal transformation is a symmetry (exercise)

$$\delta x = -y, \quad \delta y = x.$$

This is an infinitesimal rotation about the z axis. From Noether's theorem, the corresponding conserved quantity is

$$Q = (mv_x)(-y) + (mv_y)(x) = (\vec{r} \times \vec{p})_z,$$

which is the z component of angular momentum. As an exercise you can figure out the symmetries for the other 2 components.

The next problem gives an example where you can use Noether's theorem to obtain a constant of motion which depends explicitly upon the time.

Homework Problem

*Consider a closed system of Newtonian particles interacting pairwise by central forces. Show that this system admits a **boost** symmetry, corresponding to transformation to a new inertial reference frame moving with constant relative velocity \vec{V} , i.e., the position of each particle is transformed according to*

$$\vec{r} \rightarrow \vec{r} + \vec{V}t.$$

Use Noether's theorem to compute the corresponding conserved quantities. What is their physical interpretation?